Manifolds modeled on locally convex spaces

glaroton

November 13, 2026

Contents

1 Introduction **4**

2 Manifolds and smooth maps **7**
2.1 Manifolds . . . 7
2.2 Smooth maps . . . 8

3 Tangent bundles **10**
3.1 The tangent bundle and tangent maps . . . 10
3.2 The double tangent bundle . . . 11
3.3 Curves and velocities . . . 12

4 Vector fields and the Frobenius theorem **13**
4.1 Vector fields and derivations . . . 13
4.2 Distributions and the Frobenius theorem . . . 15

5 Flows of vector fields **16**
5.1 Local flows on Banach manifolds . . . 16

6 Manifolds with rough boundary **19**
6.1 Manifolds with rough boundary . . . 19

7 Submanifolds and immersions **23**
7.1 Embedded submanifolds . . . 23
7.2 The tangent space of a submanifold of a vector space . . . 25
7.3 Immersions . . . 25
7.4 Submanifolds given by an injective map . . . 26
7.5 Embedded and split submanifolds as immersions . . . 26
7.6 Immersions with a Banach domain . . . 27
7.7 Subtleties in infinite dimensions . . . 28

8 Differential forms **31**
8.1 Local forms . . . 31
8.2 Forms on manifolds . . . 32
8.3 The vector space of forms . . . 33
8.4 Pullbacks . . . 33
8.5 The wedge product . . . 33
8.6 Insertion and the Lie derivative . . . 34
8.7 De Rham cohomology . . . 35

8.8 The Poincaré Lemma . . . 36

9 Vector bundles **38**
9.1 Vector bundles and their connections . . . 38

10 Connections **41**
10.1 Connections and the covariant derivative . . . 41
10.2 Curvature . . . 43
10.3 Connection forms and the structure equations . . . 43
10.4 Covariant derivative along curves and geodesics . . . 44
10.5 Fields of projectors: derive and project . . . 45

11 Sprays and geodesics **48**
11.1 Sprays . . . 48
11.2 Geodesics on Banach manifolds . . . 49

12 Metric geometry **53**
12.1 The rectifiable distance . . . 53
12.2 The metric Hopf–Rinow theorem . . . 54
12.3 Asymmetric distances . . . 55

13 Metrics **57**
13.1 Finsler seminorms and two continuity conditions . . . 57
13.2 Length and distance . . . 58
13.3 Lifting curves versus lifting neighborhoods . . . 59
13.4 Bilinear metrics . . . 60
13.4.1 Riemannian metrics . . . 60
13.4.2 Fundamental decompositions and the index . . . 60
13.4.3 Pseudo-Riemannian and Lorentzian metrics . . . 61
13.5 The Levi-Civita connection . . . 61
13.6 Submanifolds: orthogonal projectors in any signature . . . 63
13.7 (M, d_b) is an interior metric space . . . 64
13.8 Bounded Finsler structures and the topology . . . 65

14 Local additions and manifolds of mappings **67**
14.1 Local additions . . . 67
14.2 Real analytic local additions . . . 69
14.3 Manifolds of continuous maps . . . 70
14.4 Manifolds of C^ℓ maps . . . 71
14.5 Smooth maps between atlases with varying models . . . 72
14.6 $C^{k,\ell}$ maps on products of manifolds . . . 73
14.7 The exponential law for manifolds of mappings . . . 73
14.8 The compact-open C^ℓ topology via iterated tangent maps . . . 74
14.9 The jet topology and Definition 4.1.2 . . . 74

15 Analytic manifolds **76**
15.1 The analytic groupoids . . . 76
15.2 Analytic maps . . . 77
15.3 Complexifications . . . 77
15.4 Real analytic forms . . . 78

16 Nijenhuis operators, complex and Kähler manifolds **79**

17 Symplectic manifolds **81**
17.1 Symplectic forms . . . 81
17.2 Hamiltonian and symplectic vector fields . . . 81
17.3 Symplectic maps . . . 82
17.4 The canonical symplectic form . . . 82
17.5 The real analytic Darboux theorem . . . 83
17.6 The cotangent bundle . . . 83
17.7 The Darboux theorem . . . 84

18 Examples **86**
18.1 Closed subspaces, open subsets and graphs . . . 86
18.2 The unit sphere of an inner product space . . . 87
18.3 The unitary group of a Banach $*$-algebra . . . 89
18.4 Smooth periodic functions and $\mathrm{Diff}^+(S^1)$ . . . 90
18.4.1 The Fréchet space C^∞_{per} . . . 90
18.4.2 The universal cover of $\mathrm{Diff}^+(S^1)$ . . . 91

19 Counterexamples **93**
19.1 Weak immersions, injective immersions and split immersions . . . 93
19.1.1 A weak immersion that is not an immersion . . . 93
19.1.2 An injective immersion that is not initial: the figure eight . . . 94
19.1.3 An immersion that is not split: $c_0 \subseteq \ell^\infty$ . . . 94
19.2 A Finsler length structure that is not a Finsler metric . . . 95

Chapter 1

Introduction

This blueprint describes the mathematics formalized in the Lean library lc-manifolds: manifolds modeled on locally convex spaces, their tangent bundles, vector fields and flows, differential forms, submanifolds and immersions, vector bundles, connections and geodesics, metrics, manifolds of mappings, analytic, complex and symplectic manifolds, and a collection of examples from functional analysis and from groups of diffeomorphisms. Every definition and theorem below is linked to its Lean counterpart, and the dependency graph shows how the results build on each other. All the statements are proved in Lean, with no `sorry`, and the main theorems depend only on the standard axioms of Lean.

The setting

The differential calculus is the one of Michal and Bastiani, formalized in the companion library lc-analysis [LCA]: a map between open subsets of locally convex spaces is C^1 if its directional derivatives exist and depend continuously on the point and on the direction, jointly. This is the calculus of infinite-dimensional Lie theory, as presented by Neeb [Nee06] and by Glöckner and Neeb [GN26], which we follow for the foundations. For the geometry of Banach manifolds (sprays, connections, Riemannian metrics) we follow Lang [Lan99] and the book [Lar26], and for metric geometry [Lar19].

A manifold modeled on E is a charted space whose transition maps are smooth diffeomorphisms between open subsets of E. Mathlib [mC20] accepts an arbitrary topological model space in its notion of charted space, so we do not need a separate theory of manifolds: we only add the groupoid of Michal–Bastiani diffeomorphisms.

Conventions

Three conventions are used throughout, and we state them once and for all.

- **Model spaces need not be complete.** The model space E is a real locally convex Hausdorff space, with no completeness assumption, following Glöckner and Neeb [GN26]. We call a manifold *Banach*, *Hilbert* or *Fréchet* only when its model is complete; a manifold modeled on a normed space that is not complete is not a Banach manifold. Whenever a result needs completeness, it is part of the statement.

- **No inverse function theorem beyond Banach spaces.** Outside Banach spaces there is no inverse function theorem, and there are no existence theorems for ordinary differential equations. On general models we avoid both: submanifolds are given by slice charts, examples are built with explicit charts, and geodesics are characterized, not constructed. Flows, geodesics, the exponential map and the Frobenius and Darboux theorems are proved for Banach models only.
- **Infinite-dimensional subtleties are flagged.** Notions that coincide in finite dimensions often split apart here: immersions and weak immersions, embedded and initial submanifolds, strong and weak metrics, continuity on the tangent bundle and continuity along curves. We keep a separate name for each notion, and each splitting is illustrated by a formalized counterexample whenever we have one.

Metrics: two questions, two words

Chapter 13 uses two pairs of words that should not be confused. For a field of Finsler seminorms μ, the question is how μ_p depends on p. A *Finsler metric* is continuous on the tangent bundle; this is what the term means without further qualification. A *Finsler length structure* is only continuous along C^1 curves: it is what one needs to measure lengths, and quotient metrics on homogeneous spaces are of this kind. Every Finsler metric is a Finsler length structure, and the converse fails already on $\mathbb{R}^2$. For a field of bilinear forms g, the question is which topology g_x induces on the tangent space: a *strong* metric induces the topology of the model, a *weak* one may induce a coarser topology. Weak Riemannian metrics are Finsler metrics; Lorentzian and pseudo-Riemannian metrics, of any index, are not.

Organization

The chapters follow the folders of the library.

- **Foundations.** Manifolds and smooth maps (Chapter 2), tangent bundles (Chapter 3), vector fields and the Frobenius theorem (Chapter 4), flows (Chapter 5), manifolds with rough boundary (Chapter 6), and submanifolds and immersions (Chapter 7).
- **Forms and bundles.** Differential forms (Chapter 8), vector bundles (Chapter 9), connections (Chapter 10), and sprays and geodesics (Chapter 11).
- **Metrics.** Metric geometry of length spaces (Chapter 12), and Finsler, Riemannian, Lorentzian and pseudo-Riemannian metrics on manifolds (Chapter 13).
- **Further structures.** Local additions and manifolds of mappings (Chapter 14), analytic manifolds (Chapter 15), complex manifolds (Chapter 16) and symplectic manifolds (Chapter 17).
- **Examples** (Chapter 18) and **counterexamples** (Chapter 19).

How the library was written

@glaroton set the mathematical program, chose the framework and the references, fixed the design constraints and reviewed the results. The Lean proofs were written by the AI assistant

Claude (Anthropic), working under his direction. Correctness does not rest on trusting either of them: Lean's kernel [dMU21] checks every proof, and the statements are kept close to the literature so that they can be compared with it directly.

Chapter 2

Manifolds and smooth maps

In this chapter we set up the basic objects of the library: manifolds modeled on locally convex spaces and C^k maps between them. The calculus is the Michal–Bastiani calculus of [LCA]; the references for the manifold theory are [Nee06, §I.3] and [GN26, Ch. 3], and [Lan99] for the Banach case. Throughout, E, F, G denote real Hausdorff locally convex spaces. We stress from the start that *no completeness is assumed*: nothing in this chapter needs it. Tangent bundles, vector fields and manifolds with rough boundary are treated in the next chapters.

2.1 Manifolds

Definition 2.1 (The Michal–Bastiani groupoid)**.** The groupoid $\mathcal{G}_{\mathrm{MB}}(E)$ consists of the homeomorphisms $\varphi\colon U \to V$ between open subsets of E such that φ is smooth on U and φ^{-1} is smooth on V, in the sense of Michal–Bastiani.

That this is a structure groupoid (closed under composition, inversion and restriction, and local in nature) follows from the chain rule and the locality of smoothness.

Definition 2.2 (Manifolds)**.** A *manifold modeled on* E is a topological space M with an atlas of charts with values in E whose transition maps belong to $\mathcal{G}_{\mathrm{MB}}(E)$ [Nee06, Definition I.3.1]. A *Banach manifold* is a manifold modeled on a normed space which is complete; a *Hilbert manifold* is one modeled on a complete inner product space.

In Lean, a manifold is a Mathlib `ChartedSpace` E M with `HasGroupoid` for $\mathcal{G}_{\mathrm{MB}}(E)$; Mathlib's charted spaces accept an arbitrary topological model space, so no new infrastructure is needed. Every chart comes with a *preferred chart* c_x = `chartAt` x at each point $x \in M$, and we use it systematically.

Remark 2.3 (The model space need not be complete). We follow the convention of Glöckner–Neeb [GN26, §5.7]: the model space of a manifold is not assumed to be complete. The formal theory of this and the following chapters (charts, smooth maps, tangent bundle, vector fields, connections, submanifolds) does not use completeness, since the calculus of [LCA] replaces integrals in the model space by Hahn–Banach arguments. Results that need completeness (integrals with values in E, ordinary differential equations, exponential maps, inverse function theorems) say so in their hypotheses. In particular, a manifold modeled on an incomplete normed space is a manifold in our sense, but it is not a Banach manifold. Without completeness things do go wrong: [GN26, Example 5.6.8] is a Lie group modeled on an incomplete normed algebra that has no exponential function.

On normed spaces, Michal–Bastiani smoothness agrees with Fréchet smoothness, so for complete normed E a Banach manifold is the classical object of [Lan99].

Proposition 2.4 (Products)**.** *If M and N are manifolds modeled on E and F, then $M \times N$, with the product charts, is a manifold modeled on $E \times F$. The projections onto M and N are smooth, and a pair (f, g) of C^k maps $P \to M$, $P \to N$ is a C^k map $P \to M \times N$.*

Proof. The transition maps of the product atlas are products $\varphi \times \psi$ of transition maps, and a product of smooth diffeomorphisms between open sets is again one, since $(\varphi \times \psi) = (\varphi \circ \pi_1, \psi \circ \pi_2)$ and pairs of smooth maps are smooth. In charts the projections are the linear maps π_1, π_2. □

In Lean the model space of the product is Mathlib's type synonym `ModelProd` E F of $E \times F$, which avoids instance diamonds when M or N is itself a product.

2.2 Smooth maps

For maps $f \colon M \to N$, open $U \subseteq M$ and charts e of M, e' of N, we denote by

$$D(f, U, e, e') = e(\operatorname{dom} e) \cap e^{-1}(U \cap f^{-1}(\operatorname{dom} e')) \subseteq E$$

the domain of the local representative $e' \circ f \circ e^{-1}$. It is open when f is continuous on U (`LocallyConvex.Manifolds.isOpen_chartDomain`).

Definition 2.5 (C^k maps between manifolds)**.** A map $f \colon M \to N$ is C^k on an open set $U \subseteq M$ if it is continuous on U and, for every $x \in U$, the local representative $c'_{f(x)} \circ f \circ c_x^{-1}$ in the preferred charts is C^k on $D(f, U, c_x, c'_{f(x)})$. It is *smooth* on U if it is C^k on U for every k.

Only the preferred charts appear in the definition. The next result says that this is no restriction.

Theorem 2.6 (Independence of the charts)**.** *If f is C^k on U, then its local representatives $e' \circ f \circ e^{-1}$ in arbitrary charts of the atlases (or of the maximal atlases) are C^k on $D(f, U, e, e')$. Conversely, a continuous map whose local representatives in the preferred charts are C^k near the center of each chart is C^k.*

Proof. The transition maps are smooth by definition of the atlas. Near a point $y = e(p)$ we write $e' f e^{-1} = (e' c'^{-1}) \circ (c' f c^{-1}) \circ (c e^{-1})$ with $c = c_p$, $c' = c'_{f(p)}$; the three factors are C^k and the chain rule applies. Since C^k is a local property, this suffices. □

Theorem 2.7 (Composition)**.** *If f is C^k on U, g is C^k on V and $f(U) \subseteq V$, then $g \circ f$ is C^k on U. Identities and constant maps are smooth.*

Proof. In the preferred charts at x and $g(f(x))$, near a point we insert the preferred chart d at the intermediate point and write the local representative of $g \circ f$ as the composition of the local representatives of g and f in the charts involving d. These are C^k by Theorem 2.6, and we conclude by the chain rule. □

Proposition 2.8 (Consistency with the calculus)**.** *For a map $f \colon E \to F$ between the model spaces themselves (with their identity charts) and $U \subseteq E$ open, f is C^k on U as a map between manifolds if and only if it is C^k on U in the sense of Michal–Bastiani.*

Proof. The preferred charts are identities, so $D(f, U, \mathrm{id}, \mathrm{id}) = U$ and the local representative is f itself. □

Remark 2.9 (Universes). The model spaces E, F, G live in a common universe, because the Michal–Bastiani derivative of a map defined on E is a map defined on $E \times E$. The manifolds themselves are unrestricted. A consequence that will reappear below is that real-valued functions on M are not treated as maps between manifolds (the model space $\mathbb{R}$ need not be in the universe of E), but through their local representatives.

Chapter 3

Tangent bundles

The tangent bundle and the tangent map of a manifold modeled on a locally convex space, the double tangent bundle with its canonical flip, and curves with their velocities [GN26, §3.3], [Lan99, Ch. III]. As in Chapter 2, no completeness is assumed.

3.1 The tangent bundle and tangent maps

Definition 3.1 (The tangent bundle)**.** For charts i, j of the atlas of M and x in both domains, the *coordinate change* of tangent vectors is $v \mapsto \mathrm{d}(j \circ i^{-1})(i(x))\, v$. The *tangent bundle* TM is the fiber bundle with fiber E obtained by gluing the trivial bundles over the chart domains along these coordinate changes. For a chart i, the map $TM \supseteq TM|_{\mathrm{dom}\, i} \to E \times E$ that sends (x, v) to $(i(x), v_i)$, with v_i the coordinates of v in the chart i, is the *chart of* TM *over* i.

In Lean, TM is the total space of a Mathlib `FiberBundleCore` indexed by the charts of the atlas, whose index at x is the preferred chart c_x. Thus a point of TM is a pair (x, v) with $x \in M$ and $v \in E$, and v *is the tangent vector written in the coordinates of the preferred chart at* x. In any other chart i, its coordinates are $\mathrm{d}(i \circ c_x^{-1})(c_x(x))\, v$. This is a choice of representative, not a restriction: every construction below (tangent maps, local representatives of vector fields, Lie brackets, derivations, velocities) is defined in the preferred chart and then shown to transform correctly in every chart of the atlas. The coordinate changes satisfy the cocycle condition by the chain rule (`LocallyConvex.Manifolds.tangentCoordChange_comp`), and they are continuous in (x, v) since the transition maps are C^1.

Theorem 3.2 (TM is a manifold)**.** *The charts of* TM *over the charts of* M *form an atlas modeled on* $E \times E$ *whose transition maps are*

$$(y, w) \longmapsto (\tau(y), \mathrm{d}\tau(y)\, w), \qquad \tau = j \circ i^{-1}.$$

In particular they are smooth, and TM *is a manifold modeled on* $E \times E$*.*

Proof. The formula is the cocycle property of the coordinate changes. The map $(y, w) \mapsto \mathrm{d}\tau(y)\, w$ is smooth because τ is, by the recursive definition of C^k, and pairs of smooth maps are smooth. The inverse transition has the same form with τ^{-1}. □

Definition 3.3 (The tangent map)**.** Let $f \colon M \to N$. The *tangent map* $Tf \colon TM \to TN$ sends (x, v) to

$$\big(f(x),\ \mathrm{d}(c'_{f(x)} \circ f \circ c_x^{-1})(c_x(x))\, v\big),$$

where c_x and $c'_{f(x)}$ are the preferred charts at x and $f(x)$.

The definition makes sense for every f, since d is a total function in Lean; it is meaningful when f is C^1.

Theorem 3.4 (The tangent map in charts; chain rule)**.** *Let $f\colon M \to N$ be C^1. The derivatives of the local representatives of f transform under changes of charts by the derivatives of the transition maps. Consequently, in the charts of TM and TN over charts i of M and j of N, the tangent map reads*

$$(y, w) \longmapsto (h(y), \mathrm{d}h(y)\, w), \qquad h = j \circ f \circ i^{-1}.$$

Moreover $T\mathrm{id} = \mathrm{id}$ and, for C^1 maps f and g, $T(g \circ f) = Tg \circ Tf$.

Proof. Near a point, the local representative of f in one pair of charts agrees with the composition of its local representative in another pair with transition maps; the chain rule for maps that agree with a composition near the point gives the transformation laws. The formula in charts follows by composing these laws with the coordinate changes of TM and TN, and functoriality is the chain rule in the preferred charts. □

Theorem 3.5 (Smoothness of the tangent map)**.** *If $f\colon M \to N$ is C^{k+1}, then $Tf\colon TM \to TN$ is C^k.*

Proof. In the charts of the tangent bundles Tf is $(y, w) \mapsto (h(y), \mathrm{d}h(y)\, w)$, and $\mathrm{d}h$ is C^k on its domain because h is C^{k+1}. □

3.2 The double tangent bundle

The double tangent bundle is $TTM = T(TM)$, the tangent bundle of the manifold TM (Theorem 3.2). We follow [Lar26, Cap. 5] and [Lan99, §IV.3]. A chart i of M gives a chart of TM and then a chart of TTM, in which points are written $((x, v), (u, w))$: (x, v) is the base point in TM, and (u, w) the tangent vector at it.

Theorem 3.6 (Transition maps of TTM)**.** *Let $\tau = j \circ i^{-1}$. In the charts of TTM over i and j, the transition map is*

$$((x, v), (u, w)) \longmapsto \big((\tau x, \mathrm{d}\tau_x v),\ (\mathrm{d}\tau_x u,\ \mathrm{d}^2\tau_x(u, v) + \mathrm{d}\tau_x w)\big).$$

Proof. The transition maps of TTM are those of the tangent bundle of TM, namely the transition map $\sigma(x, v) = (\tau x, \mathrm{d}\tau_x v)$ of TM together with its derivative. Differentiating σ in the direction (u, w) gives $(\mathrm{d}\tau_x u, \mathrm{d}^2\tau_x(u, v) + \mathrm{d}\tau_x w)$, since $\mathrm{d}\tau_x$ is linear in v. □

Theorem 3.7 (The canonical flip)**.** *The swap $((x, v), (u, w)) \mapsto ((x, u), (v, w))$ commutes with the transition maps of TTM. Hence the map $\kappa\colon TTM \to TTM$ defined as the swap in the preferred chart reads as the swap in every chart of TTM over a chart of M. It covers the identity of M, and $\kappa \circ \kappa = \mathrm{id}$.*

Proof. By Theorem 3.6, the claim amounts to $\mathrm{d}^2\tau_x(u, v) = \mathrm{d}^2\tau_x(v, u)$, the symmetry of the second differential. The rest is immediate from the coordinate expression. □

The smoothness of κ is not stated in the library.

3.3 Curves and velocities

Curves are maps $\alpha\colon \mathbb{R} \to M$. We treat them, like functions, through their local representatives, without regarding $\mathbb{R}$ as a manifold.

Definition 3.8 (C^k curves and lifts)**.** A curve $\alpha\colon \mathbb{R} \to M$ is C^k if it is continuous and $i \circ \alpha$ is C^k on the open set $\alpha^{-1}(\operatorname{dom} i)$ for every chart i of the atlas. A *lift* of α is a curve $\beta\colon \mathbb{R} \to TM$ with $\beta(t) \in T_{\alpha(t)}M$; its fiber coordinate in the chart i is the second component $\mu_i(t)$ of the chart of TM over i at $\beta(t)$. The lift is C^k if every μ_i is C^k on $\alpha^{-1}(\operatorname{dom} i)$.

Definition 3.9 (Velocity)**.** The *velocity* of a curve α is the lift $\dot\alpha(t) = (\alpha(t), (c_{\alpha(t)} \circ \alpha)'(t))$, the derivative of the local representative in the preferred chart at $\alpha(t)$.

Theorem 3.10 (The velocity in charts)**.** *Let α be a C^1 curve. For charts i, j around $\alpha(t)$ we have $(j \circ \alpha)'(t) = \mathrm{d}(j \circ i^{-1})(i(\alpha(t)))\,(i \circ \alpha)'(t)$; hence the fiber coordinate of $\dot\alpha(t)$ in any chart j is $(j \circ \alpha)'(t)$. The velocity is a continuous curve in TM, and if α is C^{k+1}, then $\dot\alpha$ is a C^k lift.*

Proof. Near t, $j \circ \alpha = (j \circ i^{-1}) \circ (i \circ \alpha)$, and we apply the chain rule. The second statement is the first one with i the preferred chart. In the chart of TM over i, $\dot\alpha$ reads $s \mapsto (i(\alpha(s)), (i \circ \alpha)'(s))$, which is continuous (respectively C^k in the second component); the charts of TM are homeomorphisms onto their images. □

Proposition 3.11 (Germs and the chain rule for velocities)**.** *The velocity $\dot\alpha(t)$ only depends on the germ of α at t. If $p\colon N \to M$ is C^1 and Γ is a C^1 curve in N, then $Tp\,(\dot\Gamma(t))$ is the velocity of $p \circ \Gamma$ at t.*

Proof. Directional derivatives only depend on germs. For the chain rule, near t we write $c' \circ p \circ \Gamma = (c' \circ p \circ c^{-1}) \circ (c \circ \Gamma)$ in the preferred charts c at $\Gamma(t)$ and c' at $p(\Gamma(t))$, and apply the chain rule. □

Remark 3.12. The continuity of $\dot\alpha$ in TM is what makes lengths of curves meaningful for Finsler metrics (see Definition 13.4); the chain rule for velocities is used for quotient metrics, where curves in N are pushed down along a projection p. On the other hand, we do not claim the existence of integral curves of vector fields, or of curves with prescribed velocity: this would require existence theorems for ordinary differential equations, which fail in general locally convex spaces, and in Banach spaces need completeness.

Chapter 4

Vector fields and the Frobenius theorem

Vector fields, their Lie bracket and their action on functions as derivations, and the Frobenius theorem on Banach manifolds. The bracket is defined in charts and needs no completeness; the Frobenius theorem needs Banach models, and it fails already for distributions of rank one on a Fréchet space (see the counterexamples of [LCA]). The local theorem is the one of lc-analysis, after Penot [Pen70].

4.1 Vector fields and derivations

Definition 4.1 (Vector fields)**.** A *vector field* on M is a section $X\colon M \to TM$ of the tangent bundle. Its *local representative* in a chart i is $\widetilde{X}_i\colon i(\operatorname{dom} i) \to E$, $\widetilde{X}_i(y) =$ the coordinates of $X(i^{-1}y)$ in the chart i.

Proposition 4.2 (Local representatives)**.** *On chart overlaps, $\widetilde{X}_j \circ \tau = \mathrm{d}\tau(\cdot)\,\widetilde{X}_i$ for the transition map $\tau = j \circ i^{-1}$. A vector field is C^k as a map $M \to TM$ if and only if its local representatives are C^k (one direction in every chart of the atlas, the other from the preferred charts).*

Proof. The transformation law is the cocycle property of the coordinate changes. The local representative $\widetilde{X}_i$ is the second component of the local representative of X in the charts i and Ti, which gives both implications. □

For maps X, Y defined on an open subset of E, the *local bracket* is

$$[X,Y](y) = \mathrm{d}Y(y)\,X(y) - \mathrm{d}X(y)\,Y(y).$$

Lemma 4.3 (The local bracket)**.** *Let τ be C^2, and let X_i, Y_i, X_j, Y_j be C^1 with $X_j \circ \tau = \mathrm{d}\tau \cdot X_i$ and $Y_j \circ \tau = \mathrm{d}\tau \cdot Y_i$ near y. Then $[X_j, Y_j](\tau y) = \mathrm{d}\tau(y)\,[X_i, Y_i](y)$. The local bracket satisfies the Jacobi identity for C^2 fields, and it is C^k for C^{k+1} fields.*

Proof. Differentiating $y \mapsto \mathrm{d}\tau(y)\,X_i(y)$ in the direction $Y_i(y)$ gives $\mathrm{d}^2\tau(y)(Y_i, X_i) + \mathrm{d}\tau(y)\,\mathrm{d}X_i(y)\,Y_i$. In the difference defining the bracket the second derivatives $\mathrm{d}^2\tau(y)(Y_i, X_i)$ and $\mathrm{d}^2\tau(y)(X_i, Y_i)$ cancel by the symmetry of the second differential. The Jacobi identity is a direct computation with the derivative of the local bracket, where again the second derivatives cancel by symmetry. □

Definition 4.4 (The Lie bracket)**.** The *Lie bracket* of the vector fields X and Y is the vector field whose value at x is the local bracket $[\widetilde{X},\widetilde{Y}](c_x(x))$ of the local representatives in the preferred chart at x.

Theorem 4.5 (The bracket is well defined)**.** *Let X, Y be C^1 vector fields. In every chart j of the atlas, the local representative of $[X,Y]$ is the local bracket $[\widetilde{X}_j,\widetilde{Y}_j]$. The bracket is antisymmetric, it satisfies the Jacobi identity for C^2 fields, and if X and Y are C^{k+1}, then $[X,Y]$ is C^k.*

Proof. The local representatives in the preferred chart at x and in j are related by Proposition 4.2, so Lemma 4.3 applies. The other statements are then read in a single chart. □

Remark 4.6 (No flows). In finite dimensions, chart independence of the bracket is usually derived from its description as the Lie derivative along the flow of X. In infinite dimensions vector fields need not have flows, so this argument is not available. We prove it directly, as above: the second derivatives of the transition maps cancel by the symmetry of second derivatives.

Theorem 4.7 (Related fields)**.** *Let $f\colon M \to N$ be C^2. We say that X on M and X' on N are f-related if $Tf \circ X = X' \circ f$. If X, Y are f-related to X', Y' (all of them C^1), then $[X,Y]$ is f-related to $[X',Y']$ [Lan99, §V.1].*

Proof. In charts, $\widetilde{X}'_j \circ h = \mathrm{d}h \cdot \widetilde{X}_i$ with $h = j \circ f \circ i^{-1}$. This is the hypothesis of Lemma 4.3 with h in place of τ, whose proof only uses that τ is C^2. □

We now let vector fields act on functions. As explained in Remark 2.9, a function $\varphi\colon M \to \mathbb{R}$ is said to be C^k when all its local representatives $\varphi \circ i^{-1}$ are C^k on $i(\operatorname{dom} i)$.

Definition 4.8 (Derivative along a vector field)**.** For a vector field X and $\varphi\colon M \to \mathbb{R}$, we let $X(\varphi)(x) = \mathrm{d}(\varphi \circ c_x^{-1})(c_x(x))\,\widetilde{X}(c_x(x))$, computed in the preferred chart at x. The product φY of a function and a vector field is $(\varphi Y)(x) = \varphi(x)\,Y(x)$.

Theorem 4.9 (Vector fields as derivations)**.** *Let φ, ψ be C^1 functions. Then $X(\varphi)(x) = \mathrm{d}(\varphi \circ j^{-1})(j(x))\,\widetilde{X}_j(j(x))$ in every chart j around x, and the Leibniz rules*

$$X(\varphi\psi) = X(\varphi)\,\psi + \varphi\,X(\psi), \qquad [X,\varphi Y] = X(\varphi)\,Y + \varphi\,[X,Y]$$

hold, the second one for C^1 vector fields X, Y.

Proof. Chart independence follows from $\widetilde{X}_j \circ \tau = \mathrm{d}\tau \cdot \widetilde{X}_i$ and the chain rule. The first Leibniz rule is the product rule in a chart. For the second one, $\widetilde{(\varphi Y)}_i = (\varphi \circ i^{-1})\,\widetilde{Y}_i$, and we differentiate this product in the local bracket. □

Theorem 4.10 (Vector fields are determined by local functions)**.** *Let c be a chart around x. For $\lambda \in E'$ we have $X(\lambda \circ c)(x) = \lambda(\widetilde{X}_c(c(x)))$. Consequently, if $X(\lambda \circ c)(x) = Y(\lambda \circ c)(x)$ for every continuous linear functional λ, then $X(x) = Y(x)$.*

Proof. The first statement is Theorem 4.9 in the chart c, since λ is linear. The second follows because continuous linear functionals separate the points of the Hausdorff locally convex space E (Hahn–Banach). □

Remark 4.11 (No bump functions). Smooth bump functions need not exist on manifolds modeled on locally convex spaces. Hence a vector field is not determined in general by its action on *globally* defined functions, and the classical identification of vector fields with derivations of $C^\infty(M)$ is not available. Theorem 4.10 is the substitute: functions defined on a chart domain suffice.

4.2 Distributions and the Frobenius theorem

Definition 4.12 (Distributions)**.** A *distribution* D on a manifold M modeled on a normed space E assigns to each $x \in M$ a subspace $D_x \subseteq T_xM$. It is *smooth and split of type* (E_1, E_2) if around each x_0, read in the preferred chart i at x_0 and through a linear isomorphism $\ell\colon E \to E_1 \times E_2$, it is the field of graphs of a smooth map $f\colon \ell(W) \to L(E_1, E_2)$: this is the local form of a split subbundle of TM [Lan99, Ch. VI, §1, pp. 155–157]. It is *involutive* if $[X, Y](x) \in D_x$ for all vector fields X, Y which are C^1 near x and tangent to D near x.

Remark 4.13. On a Banach manifold a local vector field need not extend to a global one, since there need not be smooth bump functions. Involutivity is therefore required for fields that are C^1 only near the point; asking it only for global fields would be a weaker condition.

Lemma 4.14 (Involutivity gives (Frob))**.** *If D is involutive and, near x_0, the field of graphs of f as above, then f satisfies Penot's condition (Frob) on $\ell(W)$.*

Proof. The fields reading $\ell^{-1}(a, f(\ell y)\, a)$ in the chart i, extended by 0 outside its domain, are tangent to D and C^1 near every point of the domain. Their bracket at x is computed in the preferred chart at x and transported to i by the transformation rule of the local bracket. On normed spaces the local bracket is the Lie bracket, which commutes with ℓ, and the bracket of the graph fields lies in the graph exactly under (Frob) (lc-analysis, `lieBracket_mem_graph_iff`). □

Theorem 4.15 (The Frobenius theorem on Banach manifolds [Lan99, Ch. VI, Theorems 1.1 and 1.2])**.** *Let E_1, E_2 be Banach spaces and D a smooth, split, involutive distribution of type (E_1, E_2). Then around every $x_0 \in M$ there is a chart φ of the maximal atlas, with $\varphi(x_0) = i(x_0)$, in which D is the constant subspace $\ell^{-1}(E_1 \times 0)$.*

Proof. By Lemma 4.14 and the Frobenius theorem in chart form (lc-analysis, `exists_frobenius_chart`) there is a diffeomorphism $\Phi(x, y) = (x, \alpha(x, y))$ fixing $\ell(i(x_0))$ with $D\Phi(q)(a, 0) = (a, f(\Phi q)\, a)$. Put $\psi = \ell^{-1} \circ \Phi \circ \ell$ and $\varphi = \psi^{-1} \circ i$. Since $D\Phi(q)$ is injective and preserves first components, $D\Phi(q)\, v \in \operatorname{graph} f(\Phi q)$ if and only if $v \in E_1 \times 0$. □

Theorem 4.16 (Converse of Frobenius)**.** *If around every point there is a chart of the maximal atlas in which D is a constant subspace $\ell^{-1}(E_1 \times 0)$, then D is involutive. Hence a smooth split distribution with E_1, E_2 Banach is involutive if and only if it admits such flat charts.*

Proof. Read in a flat chart, two vector fields tangent to D near x take values in the closed subspace $\ell^{-1}(E_1 \times 0)$, hence so do their derivatives and their local bracket. The local bracket transforms like a vector field under the change of charts, so the bracket in the preferred chart at x lies in D_x. No completeness or splitting is needed for this direction. □

Chapter 5

Flows of vector fields

Integral curves and flows of smooth vector fields on manifolds modeled on Banach spaces: uniqueness, local flows, the maximal flow, the escape lemma and a completeness criterion, after Lang [Lan99, Ch. IV] and [GN26, §2.5], and the real analytic case. Beyond Banach spaces even linear equations can have no solution, or more than one, so this chapter is restricted to Banach models.

5.1 Local flows on Banach manifolds

On manifolds modeled on locally convex spaces vector fields need not have local flows, nor unique integral curves [GN26, Examples 2.4.2 and 2.4.3], so Theorem 8.39 assumes a local flow. On Hausdorff manifolds modeled on Banach spaces it exists, and Theorem 8.39 holds unconditionally.

Lemma 5.1 (Integral curves and chart changes)**.** *Let M be modeled on a Banach space. If a curve c is continuous at t and $(\varphi_i \circ c)'(t) = \tilde{X}_i(\varphi_i(c(t)))$ in one chart i around $c(t)$, then the same holds in every chart around $c(t)$.*

Proof. The chain rule for the transition map, which is Fréchet differentiable, and the change of charts law $\tilde{X}_j \circ \tau = \mathrm{d}\tau \cdot \tilde{X}_i$ for $\tau = \varphi_j \circ \varphi_i^{-1}$. □

Theorem 5.2 (Uniqueness of integral curves [Lan99, Theorem IV.2.1])**.** *Let M be Hausdorff and modeled on a Banach space. Two integral curves of a smooth vector field on an open connected set I of times that agree at one instant agree on I.*

Proof. In a chart both curves solve $c' = \tilde{X}_i \circ c$, whose right hand side is locally Lipschitz, so they agree near every instant where they meet; since M is Hausdorff the set of such instants is closed in I. □

Theorem 5.3 (Local flows on Banach manifolds)**.** *Let M be Hausdorff and modeled on a Banach space, and X a smooth vector field. Around each point there are $\varepsilon > 0$, an open set V and a smooth map $\varphi\colon (-\varepsilon, \varepsilon) \times V \to M$ whose curves $t \mapsto \varphi(t, x)$ are integral curves with $\varphi(0, x) = x$ (a* flow box*). Gluing chosen flow boxes gives an open set $\Omega \supseteq \{0\} \times M$ of $\mathbb{R} \times M$, whose slices $\{t : (t, x) \in \Omega\}$ are intervals, and a smooth map $\mathrm{Fl}\colon \Omega \to M$ which is a local flow of X in the sense of Theorem 8.39, with $\mathrm{Fl}(s+t, x) = \mathrm{Fl}(s, \mathrm{Fl}(t, x))$ whenever (t, x), $(s, \mathrm{Fl}(t, x))$ and $(s+t, x)$ lie in Ω [GN26, Corollary 2.5.21], [Lan99, Theorem IV.2.2].*

Proof. In a chart, the local flows of lc-analysis (Fréchet C^∞ in time and initial value, from the Picard–Lindelöf theorem) give the flow boxes. By Theorem 5.2 two flow boxes agree where

both are defined, so they glue to Fl on the union Ω of the boxes. The group law holds because $s \mapsto \mathrm{Fl}(s+t,x)$ and $s \mapsto \mathrm{Fl}(s,\mathrm{Fl}(t,x))$ are integral curves on an interval containing 0 and s with the same initial value. Here Ω is not the maximal domain; see Theorem 5.4. □

Theorem 5.4 (The maximal flow [Lan99, Theorems IV.2.2 and IV.2.6], [GN26, Theorem 2.5.18, Corollary 2.5.21])**.** *Let M be Hausdorff and modeled on a Banach space, and X a smooth vector field. The integral curves of X through x glue to a* maximal integral curve $t \mapsto \mathrm{Fl}(t,x)$*, defined on an open interval $J_x \ni 0$: every integral curve through x on an open interval $I \ni 0$ is its restriction to I. Let $D = \{(t,x) : t \in J_x\}$. Then:*

1. *(group law) if $t \in J_x$, then $J_{\mathrm{Fl}(t,x)} = J_x - t$, and $\mathrm{Fl}(s+t,x) = \mathrm{Fl}(s,\mathrm{Fl}(t,x))$ whenever $(t,x) \in D$ and $(s,\mathrm{Fl}(t,x)) \in D$, in which case $(s+t,x) \in D$;*
2. *D is open in $\mathbb{R} \times M$ and* Fl *is smooth on D;*
3. *Fl is a local flow of X on D, so $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \mathrm{Fl}_t^* \omega = L_X \omega$ (Theorem 8.39) holds for it;*
4. *the flow of Theorem 5.3 is the restriction of* Fl *to its domain $\Omega \subseteq D$.*

Proof. Maximality and the group law follow from uniqueness, as in [Lan99, Theorem IV.2.2]. For openness and smoothness we follow [Lan99, Theorem IV.2.6] and [GN26, Theorem 2.5.18]: for a fixed x_0, the set of $t \in J_{x_0}$ around which D contains a neighborhood of (t,x_0) on which Fl is smooth is open, contains 0, and is closed in J_{x_0}, because near t the flow is $\mathrm{Fl}(\tau,x) = \varphi(\tau - t_1, \mathrm{Fl}(t_1,x))$ for a flow box φ at $\mathrm{Fl}(t,x_0)$ and t_1 close to t. No compactness argument is needed. □

Theorem 5.5 (Escape lemma [Lan99, Theorem IV.2.3])**.** *If the forward maximal time $b = \sup J_x$ is finite, then $t \mapsto \mathrm{Fl}(t,x)$ has no cluster point as $t \to b^-$; in particular it eventually leaves every compact set. The same holds backwards.*

Proof. If q were a cluster point, a flow box at q with time ε would extend the integral curve beyond b from a time $t_1 > b - \varepsilon$ at which the curve is in the box, against maximality. Lang argues with sequences; since M need not be first countable we use cluster points of filters. We do not claim this argument is new. □

Remark 5.6 (Hypotheses). Completeness of the model gives the flow boxes and local uniqueness; on manifolds modeled on Fréchet spaces smooth vector fields need neither have integral curves nor unique ones [GN26, Examples 2.4.2 and 2.4.3]. The Hausdorff property is needed: on the line with two origins the constant field $\partial/\partial t$ has two integral curves through -1 which agree up to time 1 and then split, so there is no maximal integral curve.

Theorem 5.7 (Real analytic maximal flows [GN26, Theorem 2.5.18(b), Corollary 2.5.21])**.** *Let M be a Hausdorff real analytic manifold modeled on a real Banach space and X a vector field whose local representatives in the charts of the atlas are real analytic. Then X has real analytic flow boxes around every point, and the maximal flow* Fl$\colon D \to M$ *is real analytic on its open domain $D \subseteq \mathbb{R} \times M$.*

Proof. The flow boxes are the real analytic local flows of the local representatives on Banach spaces (lc-analysis, after [GN26, Theorem 2.4.34]), read in a chart; charts and their inverses are real analytic. Then argue as for Theorem 5.4, following [GN26, Theorem 2.5.18]: the times $t \in J_{x_0}$ near which the flow is real analytic form an open subset of J_{x_0} containing 0, closed in J_{x_0} because near t the flow is $\mathrm{Fl}(\tau,x) = \varphi(\tau - t_1, \mathrm{Fl}(t_1,x))$ for a real analytic flow box φ, and compositions of real analytic maps are real analytic. □

Theorem 5.8 (Completeness criterion [Lan99, Proposition IV.2.5])**.** *Let X be a smooth vector field on a Hausdorff manifold M modeled on a Banach space E, and let $a > 0$ and K be such that every $x \in M$ admits a chart φ at x whose target contains the ball $B(\varphi(x), a)$, on which the local representative of X is bounded by K. Then $J_x \supseteq (-a/(K+1), a/(K+1))$ for every x, and the maximal flow of X is defined on all of $\mathbb{R} \times M$.*

Proof. Fix x and $\delta = a/(K+1)$. As long as the maximal curve stays in the ball, its representative $\gamma = \varphi \circ \mathrm{Fl}(\cdot, x)$ is K-Lipschitz, so $\|\gamma(t) - \varphi(x)\| \leq K\delta < a$ for $|t| < \delta$. A connectedness argument on $(-\delta, \delta)$ shows that the curve is defined and stays in the ball there: at a limit time t the representative is Cauchy, so it converges (here E is complete) to a point of the ball, whose preimage is a cluster point of the curve; by the escape lemma, $t \in J_x$. Then the group law gives $J_x + (-\delta, \delta) \subseteq J_x$, so $J_x = \mathbb{R}$. Lang deduces the uniform existence time from the local existence theorem with explicit constants [Lan99, Proposition IV.1.1], which uses a Lipschitz constant and hence a bound on the derivative of the local representative; this argument does not need it. We do not claim it is new. The uniform radius a cannot be dropped: the constant field 1 on the interval $(0, 1)$ is bounded but not complete. □

Chapter 6

Manifolds with rough boundary

Manifolds with rough boundary, after [GN26, §3.5]: the charts take values in locally convex subsets with dense interior of the model space, with the calculus of such sets developed in [LCA]. This includes manifolds with boundary and with corners. We prove that the boundary is well defined, and treat open subsets, graphs, subspaces and the tangent bundle.

6.1 Manifolds with rough boundary

Following [GN26, §3.5], manifolds may be modeled on subsets of E that are not open: a subset $V \subseteq E$ is *locally convex* if every point of V has a convex neighborhood in V, and it has *dense interior* if $V \subseteq \overline{V^0}$. The C^k calculus of [LCA] ([GN26, §1.4]) works on such sets: $f\colon V \to F$ is C^1 if it is continuous, C^1 on V^0, and $\mathrm{d}(f|_{V^0})$ extends continuously to $V \times E$. Below, $r \in \mathbb{N}_0 \cup \{\infty\}$ and "C^r on V" means C^k on V for every $k \le r$. Mathlib's charted spaces need open targets, so rough atlases are a separate structure, built on Mathlib's partial homeomorphisms.

Definition 6.1 (Rough charts and rough atlases)**.** A *rough E-chart* of a topological space M is a homeomorphism $\varphi\colon U_\varphi \to V_\varphi$ from an open subset of M onto a locally convex subset of E with dense interior. A *C^r atlas of rough charts* is a set $\mathcal{A}$ of rough charts whose domains cover M and whose transition maps $\psi \circ \varphi^{-1}\colon \varphi(U_\varphi \cap U_\psi) \to E$ are C^r; for $r = 0$ one requires moreover that $\varphi(x) \in V_\varphi^0$ if and only if $\psi(x) \in V_\psi^0$ [GN26, Definition 3.5.1(a), p. 278].

The domains $\varphi(U_\varphi \cap U_\psi)$ are relatively open in V_φ, hence again locally convex with dense interior, so the C^r condition makes sense [GN26, Remark 3.5.2, p. 279].

Theorem 6.2 (The boundary is well defined)**.** *Let φ, ψ be charts of a C^r atlas of rough charts at $x \in M$. Then $\varphi(x) \in V_\varphi^0$ if and only if $\psi(x) \in V_\psi^0$ [GN26, Lemma 3.5.6, pp. 279–280]. Hence the* boundary *∂M, the set of points x with $\varphi(x) \notin V_\varphi^0$ for some chart φ at x [GN26, Definition 3.5.1(c), p. 278], can be tested in any chart.*

Proof. For $r = 0$ this is an axiom. Let $r \ge 1$, $\kappa = \psi \circ \varphi^{-1}$, $y = \varphi(x) \in V_\varphi^0$, and suppose $z = \kappa(y)$ is not interior. Take an open $W \ni z$ such that $C = V_\psi \cap W$ is convex with nonempty interior. By Hahn–Banach there is $\lambda \in E'$ with $\lambda < \lambda(z)$ on C^0, hence $\lambda \le \lambda(z)$ on C. Pick $a \in C^0$ and $v = z - a$, so $\lambda(v) > 0$. The chain rule applied to $\kappa \circ \kappa^{-1} = \mathrm{id}$ at z gives u with $\mathrm{d}\kappa(y, u) = v$. For $t > 0$ small, $\kappa(y + tu) \in C$, so $t^{-1}(\lambda(\kappa(y + tu)) - \lambda(z)) \le 0$; these quotients converge to $\lambda(v) > 0$, a contradiction. □

Remark 6.3 (Infinite-dimensional subtleties). For $r = 0$ the axiom on interior points cannot be dropped in infinite dimensions: a closed ball of an infinite-dimensional Banach space is homeomorphic to the whole space [GN26, Remark 3.5.9, p. 280]. The proof above uses Hahn–Banach, hence the local convexity of E and of V_ψ, but no completeness. The boundary ∂M is only a subset of M; it need not be a submanifold (corners of $[0,1]^2$), and no structure on it is constructed.

Definition 6.4 (Manifolds with rough boundary)**.** A *C^r manifold with rough boundary* is a topological space with a maximal C^r atlas of rough charts [GN26, Definition 3.5.1(b), p. 278]. Every rough atlas is contained in exactly one maximal atlas, the set of rough charts compatible with it, and the boundary does not change when the atlas is replaced by its maximal atlas.

Proof. The transition map between two charts compatible with $\mathcal{A}$ factors, near each point, through a chart of $\mathcal{A}$, and is C^r by the chain rule. □

Definition 6.5 (C^s maps)**.** A map $f\colon M \to N$ between manifolds with rough boundary is C^s if it is continuous and all its local representatives $\psi \circ f \circ \varphi^{-1}\colon \varphi(f^{-1}(U_\psi) \cap U_\varphi) \to F$ are C^s [GN26, Definition 3.5.11, p. 281].

Proposition 6.6 (Composition and chart independence)**.** *Compositions of C^s maps are C^s [GN26, Proposition 3.5.12, p. 281]. For $s \le r$, a map is C^s for given atlases if and only if it is C^s for their maximal atlases.*

Proof. Near a point, $\psi \circ (g \circ f) \circ \varphi^{-1} = (\psi \circ g \circ \kappa^{-1}) \circ (\kappa \circ f \circ \varphi^{-1})$ for a chart κ at the image point, and the chain rule on locally convex subsets with dense interior applies, since the domain of $\psi \circ g \circ \kappa^{-1}$ is relatively open in V_κ. Chart independence follows by composing with the identity maps $(M, \mathcal{A}_{\max}) \to (M, \mathcal{A})$ and $(N, \mathcal{B}) \to (N, \mathcal{B}_{\max})$, which are C^s by definition of the maximal atlases. □

Proposition 6.7 (Examples)**.**

1. *A manifold modeled on E is a C^r manifold with rough boundary, for every r, with empty boundary [GN26, Example 3.5.3(a), p. 279; Remark 3.5.7, p. 280].*
2. *Products of C^r manifolds with rough boundary, with the product charts, are C^r manifolds with rough boundary [GN26, Example 3.5.5, p. 279], and $\partial(M \times N) = \partial M \times N \cup M \times \partial N$.*
3. *A nonempty locally convex subset $U \subseteq E$ with dense interior, with the single chart $U \hookrightarrow E$, is a C^r manifold with rough boundary, whose boundary is $U \setminus U^0$ [GN26, Example 3.5.3(b), p. 279; Remark 3.5.7, p. 280]. This applies to convex sets with nonempty interior, such as closed half-spaces $\{\lambda \ge 0\}$, $\lambda \ne 0$, and $[0,1]$, whose boundary is $\{0,1\}$.*

Remark 6.8. [GN26, Example 3.5.3(b)] is stated for locally convex subsets with nonempty interior; the inclusion is a rough chart only if the interior is dense, which is what we assume. For convex sets the two conditions agree.

Proposition 6.9 (Open subsets and full submanifolds)**.** *A nonempty open subset U of a C^r manifold with rough boundary M is one, with the restricted charts, and $\partial U = U \cap \partial M$ [GN26, Example 3.5.4, p. 279]. A nonempty subset $N \subseteq M$ such that every point of N has a chart φ with $\varphi(U_\varphi \cap N)$ locally convex with dense interior (a* full submanifold*) is a C^r manifold with rough boundary, with $\partial N = N \cap (\partial M \cup (N \setminus N^0))$ [GN26, Definition 3.5.19 and (3.24), p. 284].*

Proof. The targets of the restricted charts are relatively open in, or locally convex subsets with dense interior of, the targets of the charts of M, and the transition maps are restrictions of those of M. A point $\varphi(x)$ is interior to $\varphi(U_\varphi \cap N)$ if and only if it is interior to V_φ and x is interior to N, which gives the boundary and condition (A3) for $r = 0$. □

Definition 6.10 (Submanifolds of manifolds with rough boundary)**.** Let $F \subseteq E$ be a closed vector subspace. A subset $N \subseteq M$ is a submanifold modeled on F if every point of N lies in a chart φ with $\varphi(U_\varphi \cap N) = V_\varphi \cap F$ and $V_\varphi \cap F$ with dense interior relative to F (S1), and, for $r = 0$, adapted charts agree on boundary points (S2). The charts $\varphi|_{U_\varphi \cap N}$ then form a C^r atlas of rough F-charts of N [GN26, Definition 3.5.14 and Remark 3.5.16, pp. 282–283]; the transition maps are corestricted to F by [GN26, Lemma 1.4.16].

Proposition 6.11 (Graphs)**.** *Let M be a C^r manifold with rough boundary modeled on E, N a C^r manifold without boundary modeled on F, and $f\colon M \to N$ a C^r map. Every point of the graph of f lies in a chart of $M \times N$ adapted to the graph and to $E \times \{0\}$; for $r \geq 1$ the graph is a submanifold of $M \times N$ modeled on $E \times \{0\}$, with boundary $\{(x, f(x)) : x \in \partial M\}$ [GN26, Proposition 3.5.17, pp. 283–284].*

Proof. For charts φ of M and ψ of N, the chart $\theta(x, y) = (\varphi(x), \psi(y) - \psi(f(x)))$ is $g \circ (\varphi \times \psi)$ for the C^r diffeomorphism $g(z, w) = (z, w - h(z))$, $h = \psi \circ f \circ \varphi^{-1}$, so it is compatible with the product atlas. Its target is relatively open in $\varphi(U_\varphi \cap f^{-1}(U_\psi)) \times F$, because ψ has open target, and it maps the graph onto $E \times \{0\}$. For $r = 0$, G–N obtain a submanifold with prescribed boundary [GN26, 3.5.15], which we do not formalize. □

Definition 6.12 (Manifolds with corners)**.** A manifold with rough boundary modeled on $\mathbb{R}^n$ is a manifold with corners (with boundary) if its maximal atlas contains charts around every point whose ranges are open subsets of $[0, \infty[^n$ (of $\mathbb{R}^{n-1} \times [0, \infty[$) [GN26, Remark 3.5.10, p. 281]. The quadrant itself is a manifold with corners, whose boundary is the set of points with a vanishing coordinate.

Definition 6.13 (The tangent bundle of a manifold with rough boundary)**.** Let $\mathcal{A}$ be a C^r atlas of rough charts of M, $r \geq 1$. The tangent bundle TM is glued from the sets $V_\varphi \times E$, $\varphi \in \mathcal{A}$, along the maps $T(\psi \circ \varphi^{-1})\colon (y, w) \mapsto (\psi\varphi^{-1}(y), \mathrm{d}(\psi \circ \varphi^{-1})(y, w))$; its tangent charts are $T\varphi\colon TU_\varphi \to V_\varphi \times E$ [GN26, 3.5.22, p. 285; Definitions 3.3.5 and 3.3.7, p. 257]. In Lean, TM is the total space of a fiber bundle core over the chart domains, with fiber E and coordinate changes $\mathrm{d}(\psi \circ \varphi^{-1})(\varphi(x), \cdot)$.

Remark 6.14 (Fibers over boundary points). At a boundary point x, $\mathrm{d}(\psi \circ \varphi^{-1})(\varphi(x), \cdot)$ is the continuous extension of the differential from the interior, not a derivative at an interior point. It is nevertheless continuous linear [GN26, Lemma 1.4.7] and invertible, by the chain rule [GN26, Proposition 1.4.10] on the domains $\varphi(U_\varphi \cap U_\psi)$, which are locally convex with dense interior. Hence each fiber, also over ∂M, is a topological vector space isomorphic to E through every chart; it is all of E, not a cone of inward vectors.

Theorem 6.15 (TM is a manifold with rough boundary)**.** *For $r \geq 1$, the tangent charts form a C^{r-1} atlas of rough $E \times E$-charts of TM, with transition maps $T(\psi \circ \varphi^{-1})$; the boundary of TM is $\pi^{-1}(\partial M)$, and $\pi\colon TM \to M$ is C^{r-1} [GN26, 3.5.22, p. 285; Lemma 3.3.8, p. 258].*

Proof. The targets $V_\varphi \times E$ are locally convex with dense interior. If g is C^s on D, $s \geq 1$, then $(y, w) \mapsto (g(y), \mathrm{d}g(y, w))$ is C^{s-1} on $D \times E$ [GN26, Definition 3.3.3, p. 256], since $\mathrm{d}g$ is C^{s-1} and pairs of C^{s-1} maps are C^{s-1}. As $(V_\varphi \times E)^0 = V_\varphi^0 \times E$, interior points of TM are the vectors over interior points of M, which gives the boundary of TM and, for $r = 1$, the axiom on interior points. □

Theorem 6.16 (Tangent maps and the chain rule)**.** *Let $f\colon M \to N$ be C^s, $s \geq 1$. The tangent map Tf, given in tangent charts by $T(\psi \circ f \circ \varphi^{-1})$, does not depend on the charts and is C^{s-1}; moreover $T(\mathrm{id}) = \mathrm{id}$ and $T(g \circ f) = Tg \circ Tf$ for C^1 maps [GN26, Definition 3.3.11, p. 259;*

Proposition 3.3.12, p. 259; Remark 3.3.13, p. 260; 3.5.22, p. 285]. For ordinary manifolds the coordinate changes of TM are those of Definition 3.1.

Proof. For charts φ, φ' at x and ψ, ψ' at $f(x)$, both $\mathrm{d}(\psi' \circ f \circ \varphi'^{-1})(\varphi'(x), \mathrm{d}(\varphi' \circ \varphi^{-1})(\varphi(x), v))$ and $\mathrm{d}(\psi' \circ \psi^{-1})(\psi(f(x)), \mathrm{d}(\psi \circ f \circ \varphi^{-1})(\varphi(x), v))$ are equal to $\mathrm{d}(\psi' \circ f \circ \varphi^{-1})(\varphi(x), v)$, by the chain rule on the common domain $\varphi(U_\varphi \cap U_{\varphi'} \cap f^{-1}(U_\psi \cap U_{\psi'}))$, which has dense interior; the differential on a smaller domain with dense interior is the restriction. The chain rule for $T(g \circ f)$ is the same computation. We do not claim this argument is new. □

Chapter 7

Submanifolds and immersions

In finite dimensions all reasonable notions of submanifold and immersion agree, up to some topological phenomena such as the figure eight. In infinite dimensions they split apart, already for Banach models and more so for locally convex ones. In this chapter we first construct embedded submanifolds, defined by slice charts onto a closed subspace of the model, and show that they are manifolds in their own right. Then we introduce three notions of immersion (weak, plain, split) and the corresponding notions of submanifold given by an injective map (weakly immersed, immersed, initial), and we prove the implications among them. Finally we show that for a Banach domain an immersion reflects smoothness of continuous maps. The chapter closes with a list of the infinite-dimensional subtleties; some are formalized as counterexamples in Chapter 18, others are only discussed. The references are [GN26, Def. 3.1.10, Def. 3.3.22, Rem. 3.3.23, Lemma 3.3.24], [Nee06, §II.6, §IV.4], [KMW24] and [Lar26, Cap. 2].

Throughout, E and F are real locally convex Hausdorff spaces, not necessarily complete, and M, N are manifolds modeled on E and F (Definition 2.2).

7.1 Embedded submanifolds

The whole construction rests on one calculus fact: a map with values in a closed subspace is as smooth into the subspace as it is into the ambient space.

Lemma 7.1 (Corestriction to a closed subspace)**.** *Let $T \subseteq F$ be a closed subspace, D a locally convex space, $V \subseteq D$ open, and $h\colon V \to F$ a C^k map with $h(V) \subseteq T$. Then h is C^k as a map $V \to T$.*

Proof. Induction on k. The difference quotients of h lie in T, so their limit $\mathrm{d}h(x)(w)$ lies in T because T is closed; the directional derivative of the corestriction is the corestriction of $\mathrm{d}h$, and the inductive hypothesis applies to $\mathrm{d}h$ on $V \times D$. This is [GN26, Lemma 1.3.19]. □

Definition 7.2 (Slice chart)**.** Let $N \subseteq M$ and let $S \subseteq E$ be a subspace. A chart $\varphi\colon U \to E$ is a *slice chart* for N with model S if φ belongs to the maximal atlas of M and $\varphi(U \cap N) = \varphi(U) \cap S$, that is, $q \in N$ iff $\varphi(q) \in S$ for $q \in U$.

Definition 7.3 (Embedded submanifold)**.** A subset $N \subseteq M$ is an *embedded submanifold* modeled on the subspace $S \subseteq E$ if S is closed and every point of N lies in the domain of a slice chart for N with model S.

This is [GN26, Def. 3.1.10]. No complement of S is required.

Remark 7.4 (Slice charts and the maximal atlas). The slice charts must be taken in the maximal atlas, i.e. among the charts compatible with the atlas of M, and not in the atlas itself. For instance, when $M = E$ the atlas only contains the identity, which is a slice chart only for subspaces.

Definition 7.5 (Induced charts)**.** A slice chart $\varphi\colon U \to E$ induces a chart of N with values in S, namely the restriction $\varphi|_{U\cap N}\colon U \cap N \to \varphi(U) \cap S$, whose inverse is the restriction of φ^{-1} to $\varphi(U) \cap S$.

In Lean the induced chart is a total function, and a base point of N is used as a junk value outside the target of the inverse.

Lemma 7.6 (Transition maps of induced charts)**.** *If S is closed, the transition map between two induced charts is smooth, and so is its inverse; hence it belongs to the Michal–Bastiani groupoid of S.*

Proof. The transition map is the restriction to S of the transition map $\psi \circ \varphi^{-1}$ of M, which is smooth because both charts are in the maximal atlas. It takes values in S, so it is smooth into S by Lemma 7.1. The inverse is handled by symmetry. □

Theorem 7.7 (Embedded submanifolds are manifolds)**.** *Let $N \subseteq M$ be an embedded submanifold modeled on S. The induced charts form an atlas of N, with the topology induced by M, and with it N is a manifold modeled on S.*

Proof. The atlas consists of all induced charts; at each point we choose one given by the definition. Compatibility is Lemma 7.6. □

Proposition 7.8 (The inclusion is smooth)**.** *The inclusion $N \hookrightarrow M$ of an embedded submanifold is smooth.*

Proof. In an induced chart and a chart c of M, the local representative of the inclusion is $(c \circ \varphi^{-1}) \circ \iota_S$, with $\iota_S\colon S \to E$ the inclusion: a smooth transition map of M composed with a continuous linear map. □

The simplest example is a closed subspace of the model space itself.

Proposition 7.9 (Closed subspaces)**.** *A closed subspace $S \subseteq F$ is an embedded submanifold of F modeled on S, with the identity as slice chart. If S is complemented, it is a split submanifold.*

Proof. The identity of F satisfies $q \in S \iff \mathrm{id}(q) \in S$. □

Remark 7.10 (Subtleties of the definition). When S is complemented, Glöckner and Neeb call N a *split* submanifold (Definition 7.26). In the Banach literature (e.g. Bourbaki) "submanifold" usually means split, and non-split ones are called quasi-submanifolds. For locally convex models the splitting condition is much less useful [GN26, Rem. 3.1.11]. On the other hand, the usual criteria to produce submanifolds (regular values, split immersions) rest on the inverse or implicit function theorems, which are not available for general locally convex models. In this library embedded submanifolds are therefore produced by exhibiting slice charts by hand (see Chapter 18).

7.2 The tangent space of a submanifold of a vector space

When the ambient manifold is a locally convex space F itself, the tangent space of an embedded submanifold can be described as a subspace of F. This is the description used for fields of projectors and the induced connections in the chapter on connections.

Definition 7.11 (Tangent space)**.** Let $X \subseteq F$ be an embedded submanifold modeled on S and $p \in X$. Choose a slice chart φ at p. The tangent space of X at p is

$$T_pX = \{w \in F : \mathrm{d}\varphi(p)(w) \in S\}.$$

Lemma 7.12 (Independence of the chart)**.** *If φ, φ' are slice charts at $p \in X$, then $\mathrm{d}\varphi(p)(w) \in S$ iff $\mathrm{d}\varphi'(p)(w) \in S$. Hence T_pX can be read in any slice chart, and it contains $\mathrm{d}\varphi^{-1}(\varphi(p))(s)$ for every $s \in S$.*

Proof. The transition map $\tau = \varphi' \circ \varphi^{-1}$ maps the slice into the slice, so by closedness of S its derivative at $\varphi(p)$ maps S into S; the same holds for τ^{-1}. The chain rule then gives the equivalence. The last claim follows since $\mathrm{d}\varphi \circ \mathrm{d}\varphi^{-1}$ is the identity. □

Proposition 7.13 (Curves)**.** *A curve $\beta : \mathbb{R} \to X$ which is smooth as a curve in F is a C^k curve of the manifold X, for every k.*

Proof. In a slice chart φ the local representative is the corestriction to S of $\varphi \circ \beta$, which is smooth by Lemma 7.1. □

Proposition 7.14 (Tangent space of a closed subspace)**.** *For a closed subspace $S \subseteq F$, $T_pS = S$ at every $p \in S$.*

Proof. Read the tangent space in the identity chart, whose derivative is the identity. □

7.3 Immersions

Let $f : M \to N$ be smooth. We read its differential at x in the preferred charts c at x and c' at $f(x)$, as the map $v \mapsto \mathrm{d}(c' \circ f \circ c^{-1})(c(x))(v)$, $E \to F$. It is continuous and linear.

Lemma 7.15 (Derivatives of transition maps)**.** *For charts e, e' of the maximal atlas, the derivative of $e \circ e'^{-1}$ at the image point is a left inverse of the derivative of $e' \circ e^{-1}$; in particular the latter is injective. The differential of a smooth map, read in the preferred charts, is continuous.*

Proof. Differentiate $(e \circ e'^{-1}) \circ (e' \circ e^{-1}) = \mathrm{id}$ near the point, by the chain rule. □

Definition 7.16 (Weak immersion)**.** A smooth map $f : M \to N$ is a *weak immersion* if its differential is injective at every point. No condition is imposed on its range.

Definition 7.17 (Immersion)**.** A smooth map $f : M \to N$ is an *immersion* if, at every point, its differential has closed range and is a topological embedding, i.e. a topological isomorphism onto a closed subspace of F.

Definition 7.18 (Split immersion)**.** A smooth map $f : M \to N$ is a *split immersion* if, at every point, its differential has a continuous linear left inverse $r : F \to E$. Equivalently, the differential is a topological isomorphism onto a complemented subspace.

Proposition 7.19 (Split immersion ⇒ immersion ⇒ weak immersion)**.** *Every split immersion is an immersion, and every immersion is a weak immersion.*

Proof. Let D be the differential and r a continuous left inverse. The range of D is the fixed set of the continuous map $D \circ r$, hence closed, and D is an embedding since r is a continuous left inverse. The second implication is immediate: embeddings are injective. □

Remark 7.20 (Dependence on charts). The three definitions are stated in the preferred charts. Since the derivatives of transition maps are linear homeomorphisms (Lemma 7.15), the conditions do not depend on this choice; the chart independence is not stated separately in Lean.

Proposition 7.21 (Inclusion of a closed subspace)**.** *The linear inclusion $S \hookrightarrow F$ of a closed subspace is an immersion. It is a split immersion if and only if S is complemented.*

Proof. In the identity charts the differential is the inclusion itself, a topological embedding with closed range S. A continuous left inverse $r\colon F \to S$ gives the continuous projection $\iota_S \circ r$ onto S, and conversely. □

Remark 7.22 (Immersions in the sense of Glöckner–Neeb). [GN26, Def. 3.3.25(b)] calls f an immersion when it has, locally, the form of a continuous linear map with a continuous linear left inverse; this forces the range of the differential to be complemented. We do not use it as the definition of an immersion: it is much more restrictive than Definition 7.17. For Banach models, an immersion with complemented range has this local form by the inverse function theorem; for locally convex models that theorem is not available.

7.4 Submanifolds given by an injective map

An injective map $f\colon M \to N$ exhibits its image as a submanifold of N, with the manifold structure transported from M. This structure is extra data: the figure eight carries two such structures. Initial submanifolds are the exception, since their structure is unique [GN26, Lemma 3.3.24]; this uniqueness is not formalized.

Definition 7.23 (Weakly immersed and immersed submanifolds)**.** The map f exhibits its image as a *weakly immersed submanifold* if it is an injective weak immersion, and as an *immersed submanifold* if it is an injective immersion.

Definition 7.24 (Initial submanifold)**.** The map f exhibits its image as an *initial submanifold* if it is an injective weak immersion and, for every manifold P modeled on a locally convex space H and every map $g\colon P \to M$, the map g is smooth as soon as $f \circ g$ is smooth.

This is [GN26, Def. 3.3.22]: smooth maps into N with values in the image lift to smooth maps into M. The topology of M may be finer than the one induced by N [GN26, Rem. 3.3.23]. In Lean the test spaces H and P range over the universe of the model spaces.

Proposition 7.25. *Immersed submanifolds and initial submanifolds are weakly immersed.*

Proof. The first claim is Proposition 7.19; the second holds by definition. □

7.5 Embedded and split submanifolds as immersions

Definition 7.26 (Split submanifold)**.** An embedded submanifold $X \subseteq M$ modeled on S is *split* if S is complemented in E [GN26, Def. 3.1.10].

Lemma 7.27 (Differential of the inclusion)**.** *Let $X \subseteq M$ be an embedded submanifold modeled on S and $p \in X$. In the preferred charts, the differential of the inclusion at p is the restriction to S of a continuous linear automorphism L of E.*

Proof. Near the point, the local representative is $(c \circ \varphi^{-1}) \circ \iota_S$, where φ is the slice chart and c the preferred chart of M. Take L the derivative of $c \circ \varphi^{-1}$; its inverse is the derivative of $\varphi \circ c^{-1}$ by Lemma 7.15. □

Theorem 7.28 (The inclusion is an immersion)**.** *The inclusion of an embedded submanifold is an immersion; hence it exhibits X as an immersed submanifold.*

Proof. With L as in Lemma 7.27, the range $L(S) = (L^{-1})^{-1}(S)$ is closed, and $L|_S$ is an embedding since L^{-1} is a continuous left inverse. □

Theorem 7.29 (Split submanifolds)**.** *The inclusion of a split submanifold is a split immersion.*

Proof. A continuous projection of E onto S, composed with L^{-1}, is a continuous left inverse of $L|_S$. □

Theorem 7.30 (The inclusion is initial)**.** *The inclusion of an embedded submanifold exhibits it as an initial submanifold: a map into X is smooth as soon as it is smooth into M.*

Proof. In a slice chart φ, the local representative of a map $g\colon P \to X$ is the corestriction to S of the local representative of g as a map into M, read in φ. The latter is smooth, and so is the former by Lemma 7.1. □

Summing up, for a subset $X \subseteq M$ and an injective map $f\colon M \to N$,

$$\text{split} \Rightarrow \text{embedded} \Rightarrow \text{immersed and initial} \Rightarrow \text{weakly immersed}.$$

None of these implications can be reversed, and initial and immersed submanifolds are not comparable; see Section 7.7.

7.6 Immersions with a Banach domain

In this section E and F are normed spaces and E is complete, so that M is a Banach manifold; F need not be complete. Recall that a manifold modeled on an incomplete normed space is not a Banach manifold.

Theorem 7.31 (Immersions reflect smoothness)**.** *Let E be a Banach space, F a normed space, and $f\colon M \to N$ an immersion. Let P be a manifold modeled on a locally convex space H and $g\colon P \to M$ continuous. If $f \circ g$ is smooth, then g is smooth.*

Proof. Work in charts, with J the local representative of f at $g(x)$. For Banach spaces, Michal–Bastiani smooth maps are Fréchet smooth, so J' is continuous in norm. The differential $J'(y_0)$ is an embedding, hence bounded below, and a bound with half the constant persists on a neighborhood V of y_0. On the open set where the local representative γ of g lands in V, $J \circ \gamma$ is C^k, and the calculus result `LocallyConvex.Analysis.ContDiffMBOn.of_comp_of_boundedBelow` of [LCA] gives that γ is C^k: the derivative of γ is obtained as the limit of difference quotients, which are Cauchy by the lower bound. □

Corollary 7.32 (Topologically embedded immersions are initial)**.** *Let E be a Banach space and F a normed space. An injective immersion $f\colon M \to N$ which is a homeomorphism onto its image exhibits its image as an initial submanifold.*

Proof. If $f \circ g$ is smooth, it is continuous, so g is continuous because f is a topological embedding. Then g is smooth by Theorem 7.31. □

Remark 7.33 (Scope of the Banach results). No inverse function theorem is used, and no complement of the range of the differential is needed. On the other hand, the image of a non-split immersion need not have slice charts: whether such an immersion, when it is a homeomorphism onto its image, is an embedded submanifold is open. Completeness of E is used, to produce the derivative of the lift as a limit of Cauchy difference quotients; completeness of F is not needed. For Fréchet or more general locally convex model spaces the statement is open, since the lower bound of the differential need not persist near a point. Continuity of g cannot be dropped: the figure eight (Remark 7.37) is an injective immersion that is not initial, because a smooth curve in the plane can lift discontinuously.

7.7 Subtleties in infinite dimensions

We collect the phenomena that separate the notions above. For each we say whether it is formalized (in Chapter 18) or only discussed.

Remark 7.34 (Weak immersions versus immersions; formalized). The differential can be injective without having closed range, already between Banach spaces: the inclusion $\ell^1 \to \ell^2$ is continuous, linear and injective, and its range is dense and not closed. Hence it is a weak immersion and not an immersion. This is formalized (`not_isImmersion_lpInclusion`, Chapter 18).

Remark 7.35 (Immersions versus split immersions; formalized). A closed subspace need not be complemented, already in Banach spaces: by Phillips' lemma [Phi40], c_0 has no complement in ℓ^∞. So $c_0 \subseteq \ell^\infty$ is an embedded submanifold that is not split, and the inclusion is an immersion that is not a split immersion (compare Proposition 7.21). This is formalized, with Whitley's proof [Whi66] (`Phillips.not_isSplitImmersion_c0`, Chapter 18).

Remark 7.36 (The universal property alone; not formalized). The universal property of Definition 7.24 alone does not give an injective differential. By Joris' theorem [Jor82] (g^2 and g^3 smooth imply g smooth), the cusp $t \mapsto (t^2, t^3)$ has it, and its differential vanishes at 0. This is why injectivity of the differential is part of the definition [GN26, p. 336]; Karshon, Miyamoto and Watts call the universal property alone a *diffeological submanifold* [KMW24].

Remark 7.37 (Initial versus embedded; the figure eight is formalized). Initial does not mean embedded: the dense line in the torus is initial (not formalized). Embedded does not follow from injective immersion either. The figure eight $t \mapsto (\sin t, \sin 2t)$, $t \in (-\pi, \pi)$, is an injectively immersed curve that is not initial: the curve $s \mapsto (\sin s, -\sin 2s)$ is smooth in $\mathbb{R}^2$, stays in the figure eight, and crosses the origin along the other branch, so read in the parameter it jumps from π to 0 to $-\pi$. This is formalized, with the parameter $\theta = 2\arctan t$ (`Lemniscate.isImmersedSubmanifold_lemniscate` and `Lemniscate.not_isInitialSubmanifold_lemniscate`, Chapter 18).

Remark 7.38 (Mechanisms that destroy initiality). For an injective weak immersion, smooth maps into the ambient manifold that land in the image may fail to lift:

1. continuously, because two pieces of the image accumulate on each other (the figure eight; this occurs in finite dimensions and is formalized);
2. continuously, because there are uncountably many pieces ($\mathbb{R} \times \mathbb{R}_{\text{discrete}} \to \mathbb{R}^2$); this is excluded for second countable domains, which is why Glöckner and Neeb assume separability in [GN26, Thm. 9.2.15] (not formalized);
3. continuously, because the map is not even locally a topological embedding (the linear inclusion $\ell^1 \to \ell^2$; infinite dimensions only; that it is not an immersion is formalized, the failure of initiality is only discussed);

4. continuously, but not smoothly, even when the map is a homeomorphism onto its image (infinite dimensions only; not formalized). Let A act diagonally on the n-th block of $\ell^2(\mathbb{N}, \mathbb{R}^2)$ by a scalar $a_n \le e^{-n}$ and let $j\colon \ell^2 \to \ell^2 \times \mathbb{R}$, $j(x) = (Ax, \|x\|^2)$. Its differential $v \mapsto (Av, 2\langle x, v\rangle)$ is injective, with non-closed range at 0, and j is a homeomorphism onto its image (weak convergence plus convergence of norms). The curve $y(t) = \bigoplus_n n^{-2}(\cos nt, \sin nt)$ has constant norm and $A \circ y$ is smooth, so $j \circ y$ is smooth; but y is C^1 and not C^2 in ℓ^2. So j is topologically embedded, weakly immersed and not initial. In finite dimensions a weak immersion whose lifts are continuous is initial, so this phenomenon has no finite-dimensional analogue. Compare Corollary 7.32, where the closed range of the differential rules it out.

Remark 7.39 (Unitary orbits; not formalized). Let $A = i\sum_n \lambda_n p_n$ be a skew-adjoint operator on a Hilbert space $\mathcal{H}$, with distinct real λ_n and pairwise orthogonal projections p_n (of any rank) summing to 1. Its stabilizer in $U(\mathcal{H})$ is the block-diagonal group $K = \bigoplus_n U(p_n\mathcal{H})$, a split Lie subgroup via the conditional expectation $X \mapsto \sum_n p_n X p_n$. The orbit map $U(\mathcal{H})/K \to B(\mathcal{H})$, $uK \mapsto uAu^*$, is a weak immersion with differential $X \mapsto [X, A]$, and it is initial if and only if there are finitely many p_n. If there are finitely many p_n, the spectrum is finite and the orbit is a closed embedded split submanifold [BL23]; with two projections of infinite rank it is a Grassmannian. If there are infinitely many p_n, the spectrum is infinite, so $[\,\cdot\,, A]$ does not have closed range [BL23], and the orbit is not initial. Indeed, for orbits of Banach–Lie group actions with split stabilizer, initiality forces closed range: along $s \mapsto \exp(s\xi)\cdot A$ every derivative is bounded by a constant times $\|[\xi, A]\|$, and bump curves along directions with $\|[\xi, A]\|$ tiny are smooth in $B(\mathcal{H})$ while their lifts to $U(\mathcal{H})/K$ are continuous but not C^1.

Remark 7.40 (Initial does not imply immersion; not formalized). Let A be a compact injective diagonal operator on ℓ^2 and

$$j\colon \ell^2 \to \ell^2 \times (\ell^2 \otimes \ell^2) \times (\ell^2 \otimes \ell^2 \otimes \ell^2), \qquad j(x) = (Ax, x \otimes x, x \otimes x \otimes x),$$

with Hilbert tensor products. Then j is injective, its differential $v \mapsto (Av, x \otimes v + v \otimes x, \dots)$ is injective, and at 0 its range $A(\ell^2) \times 0 \times 0$ is not closed. Nevertheless j is a homeomorphism onto its image, the image is closed, and j is initial with respect to all test manifolds modeled on Fréchet spaces. The argument, which is only discussed, runs as follows. Let $f\colon P \to \ell^2$ with $j \circ f$ smooth.

1. Along a smooth curve c, each coordinate $\langle f \circ c, e\rangle$ has smooth square and cube, namely $\langle (f \circ c) \otimes (f \circ c), e \otimes e\rangle$ and its cubic analogue; so it is smooth by Joris' theorem [Jor82]. A curve in a Banach space all of whose scalar components are smooth is smooth, by uniform boundedness and Taylor's formula.

2. Smoothness along curves gives Michal–Bastiani smoothness when P is modeled on a Fréchet space. First, the directional derivatives $\mathrm{d}^k f(p)(v, \dots, v)$ are locally bounded: otherwise there are $p_m \to p_0$ and small w_m with $\|\mathrm{d}^k f(p_m)(w_m, \dots, w_m)\| > m$. With $t_m = 2^{-m}$, $r_m = 2^{-m-2}$ and a bump β supported in $[-1, 1]$, equal to 1 on $[-\frac{1}{2}, \frac{1}{2}]$, the curve $c(t) = p_0 + \sum_m \beta((t - t_m)/r_m)\,(p_m - p_0 + (t - t_m)w_m)$ is smooth if $p_m - p_0$ and w_m decay fast enough in every seminorm. Near t_m it is affine, $c(t) = p_m + (t - t_m)w_m$, so $(f \circ c)^{(k)}(t_m) = \mathrm{d}^k f(p_m)(w_m, \dots, w_m)$ is unbounded, a contradiction. Local bounds then give multilinearity (Boman's theorem on finite-dimensional subspaces) and joint continuity of the derivatives (mean value inequality).

So for this j the closed-range condition fails at 0 although j is initial, and initial and immersed submanifolds are not comparable. Whether j is initial with respect to test manifolds modeled on non-metrizable spaces is open: the argument above uses sequences.

Remark 7.41 (Initial Lie subgroups; not formalized). Glöckner and Neeb call an initial Lie subgroup *strictly initial* when the induced map of Lie algebras is injective [GN26, Def. 9.1.5], a condition that Neeb includes in the definition [Nee06]. Not every subgroup carries an initial Lie subgroup structure: $\ell^\infty \subseteq \mathbb{R}^{\mathbb{N}}$ does not [GN26, Ex. 9.1.8].

Chapter 8

Differential forms

Differential forms on manifolds modeled on locally convex spaces and the exterior derivative, following [GN26, Section 3.6]. Two features of the finite-dimensional theory are missing: there is no coordinate description of a form by components in a chart, and there are no partitions of unity, even on Banach manifolds. The definitions below only use local representatives, and all statements are local. No completeness of the model space or of the space of values is assumed.

Glöckner and Neeb define the exterior derivative through the Lie algebra differential of vector fields [GN26, Proposition 3.6.7, formula (3.29)] and show that it is given in charts by the formula (3.33) below. We take (3.33) as the definition and check directly that it does not depend on the chart: the terms with second derivatives of the transition maps cancel because these are symmetric. The same kind of computation proves $\mathrm{d} \circ \mathrm{d} = 0$, as indicated in [GN26, Remark 3.6.10].

8.1 Local forms

Let E and F be Hausdorff locally convex spaces and $U \subseteq E$ open.

Definition 8.1 (Local forms)**.** A *local F-valued n-form* on U is a map $\omega \colon U \times E^n \to F$ which is smooth in the sense of Michal–Bastiani and alternating n-linear in the last n variables.

Definition 8.2 (Local exterior derivative)**.** For a local n-form ω,

$$\mathrm{d}\omega(y)(v_0, \dots, v_n) = \sum_{i=0}^{n} (-1)^i \, \partial_{v_i} \omega(y)(v_0, \dots, \widehat{v_i}, \dots, v_n),$$

where ∂_{v_i} is the directional derivative in the base point y [GN26, (3.33)].

Proposition 8.3. *The exterior derivative of a local n-form is a local $(n+1)$-form.*

Proof. Smoothness: each term is the derivative $\mathrm{d}\omega$ composed with a continuous linear map. For fixed y, the partial derivative $(u, w) \mapsto \partial_u \omega(y)(w)$ is linear in u and alternating n-linear in w, because $\omega(z)$ is so for every z near y; then $\mathrm{d}\omega(y)$ is its alternation, which is alternating $(n+1)$-linear. □

Theorem 8.4 ($\mathrm{d} \circ \mathrm{d} = 0$, locally)**.** *For a local form ω on U, $\mathrm{d}(\mathrm{d}\omega) = 0$ on U.*

Proof. $\mathrm{d}(\mathrm{d}\omega)(y)(v_0,\dots,v_{n+1})$ is a double alternating sum of second derivatives $\partial_{v_i}\partial_{v_j}\omega(y)$, which are symmetric in i,j (Schwarz's theorem for Michal–Bastiani maps). The terms cancel in pairs [GN26, Remark 3.6.10]. □

Definition 8.5 (Local pullback)**.** For a smooth map $\tau\colon U \to V$ between open subsets of locally convex spaces and a local form ω on V, $(\tau^*\omega)(y)(v_1,\dots,v_n) = \omega(\tau y)(\mathrm{d}\tau(y)v_1,\dots,\mathrm{d}\tau(y)v_n)$. It is a local form on U.

Theorem 8.6 (Naturality, locally)**.** $\mathrm{d}(\tau^*\omega) = \tau^*(\mathrm{d}\omega)$ *on* U*.*

Proof. By the chain rule, $\partial_u(\tau^*\omega)(y)(w)$ is $\partial_{\mathrm{d}\tau(y)u}\omega(\tau y)(\mathrm{d}\tau(y)w)$ plus the sum over k of $\omega(\tau y)(\mathrm{d}\tau(y)w_1,\dots,\mathrm{d}^2\tau(y)(w_k,u$ the second part uses that $\omega(\tau y)$ is multilinear. In the alternating sum, the first terms give $\tau^*(\mathrm{d}\omega)$, and the terms with $\mathrm{d}^2\tau(y)$ cancel because $\mathrm{d}^2\tau(y)$ is symmetric. □

8.2 Forms on manifolds

Let M be a smooth manifold modeled on E. A tangent vector at x is given by its coordinates in the preferred chart at x (Definition 3.1).

Definition 8.7 (Differential forms)**.** An *F-valued n-form* ω on M assigns to each $x \in M$ an alternating n-linear map $\omega_x\colon (T_xM)^n \to F$ such that, for every chart φ of the atlas, the local representative $(y,v_1,\dots,v_n) \mapsto \omega_{\varphi^{-1}y}(T\varphi^{-1}v_1,\dots,T\varphi^{-1}v_n)$ is smooth on $\varphi(U)\times E^n$ [GN26, Definition 3.6.1]. We write $\Omega^n(M,F)$ for the space of these forms.

Lemma 8.8. *The local representatives of a form are local forms. On chart overlaps they are related by pullback along the transition map:* $\omega_i = \tau^*\omega_j$ *for* $\tau = j\circ i^{-1}$*.*

Proof. The coordinate changes of tangent vectors are the derivatives of the transition maps, and they compose by the chain rule. □

Definition 8.9 (Exterior derivative)**.** For $\omega \in \Omega^n(M,F)$, $(\mathrm{d}\omega)_x$ is the local exterior derivative of the local representative of ω in the preferred chart at x, evaluated at the image of x.

Theorem 8.10. *In every chart of the atlas, the local representative of* $\mathrm{d}\omega$ *is the local exterior derivative of the local representative of* ω*. In particular,* $\mathrm{d}\omega$ *is a smooth* $(n+1)$*-form [GN26, Proposition 3.6.7].*

Proof. Near a point, $\omega_i = \tau^*\omega_j$ with τ the transition map from the chart i to the preferred chart j at the point; apply Theorem 8.6. □

Theorem 8.11 ($\mathrm{d}\circ\mathrm{d} = 0$)**.** *For every* $\omega \in \Omega^n(M,F)$*,* $\mathrm{d}(\mathrm{d}\omega) = 0$ *[GN26, Proposition 3.6.9].*

Proof. In the preferred chart at x, the local representative of $\mathrm{d}\omega$ is d of the local representative of ω (Theorem 8.10); apply Theorem 8.4. □

Definition 8.12 (Functions, closed and exact forms)**.** A smooth function $f\colon M \to F$ is an F-valued 0-form. A form ω is *closed* if $\mathrm{d}\omega = 0$ and *exact* if $\omega = \mathrm{d}\eta$ for some form η [GN26, Definition 3.6.11]. By Theorem 8.11, exact forms are closed.

Proposition 8.13. *For a smooth function* f*,* $(\mathrm{d}f)_x(v) = \mathrm{d}(f\circ\varphi^{-1})(\varphi x)\,v$*, where* φ *is the preferred chart at* x*: the exterior derivative of* f *is its differential [GN26, Remark 3.6.12(a)].*

Proof. For $n = 0$ the alternating sum of Definition 8.2 has a single term. □

8.3 The vector space of forms

Definition 8.14 (Linear structure)**.** Sums and real multiples of forms are taken pointwise; their local representatives are the corresponding combinations of local representatives, hence smooth. This makes $\Omega^n(M,F)$ a real vector space.

Proposition 8.15. *The exterior derivative* $\mathrm{d}\colon \Omega^n(M,F) \to \Omega^{n+1}(M,F)$ *is linear.*

Proof. In the preferred chart at each point, directional derivatives are additive and homogeneous in the function. □

8.4 Pullbacks

Definition 8.16 (Pullback)**.** For a smooth map $\varphi\colon N \to M$ and $\omega \in \Omega^n(M,F)$, $(\varphi^*\omega)_x(v_1,\dots,v_n) = \omega_{\varphi(x)}(T_x\varphi\, v_1,\dots,T_x\varphi\, v_n)$ [GN26, Definition 3.6.6]. In charts, the local representative of $\varphi^*\omega$ is the local pullback of the local representative of ω along the local representative of φ; in particular $\varphi^*\omega$ is a smooth form.

Theorem 8.17 (Naturality of d)**.** *For a smooth map* $\varphi\colon N \to M$ *and* $\omega \in \Omega^n(M,F)$, $\mathrm{d}(\varphi^*\omega) = \varphi^*(\mathrm{d}\omega)$ *[GN26, Lemma 3.6.13].*

Proof. In the preferred charts at x and $\varphi(x)$, this is Theorem 8.6 for the local representative of φ. □

Proposition 8.18 (Functoriality of pullbacks)**.** *The pullback* φ^* *is linear,* $\mathrm{id}^*\omega = \omega$ *and* $\psi^*(\varphi^*\omega) = (\varphi\circ\psi)^*\omega$ *[GN26, (3.27)].*

Proof. The vectors in $\varphi^*\omega$ are images under the tangent map, so this is the chain rule $T(\varphi\circ\psi) = T\varphi\circ T\psi$ and $T\mathrm{id} = \mathrm{id}$. □

Proposition 8.19 (Composition with linear maps)**.** *For a continuous linear map* $\alpha\colon F \to G$, $(\alpha_*\omega)_x = \alpha\circ\omega_x$ *defines a* G*-valued form, and* $\alpha_*(\mathrm{d}\omega) = \mathrm{d}(\alpha_*\omega)$ *[GN26, Exercise 3.6.4].*

Proof. A continuous linear map commutes with directional derivatives. □

8.5 The wedge product

Let $\beta\colon E_1 \times E_2 \to E_3$ be a continuous bilinear map between locally convex spaces.

Definition 8.20 (Wedge product)**.** For $\omega \in \Omega^p(M,E_1)$ and $\eta \in \Omega^q(M,E_2)$,

$$(\omega\wedge\eta)_x(v_1,\dots,v_{p+q}) = \frac{1}{p!\,q!}\sum_{\sigma\in S_{p+q}} \operatorname{sgn}(\sigma)\,\beta\big(\omega_x(v_{\sigma(1)},\dots,v_{\sigma(p)}),\eta_x(v_{\sigma(p+1)},\dots,v_{\sigma(p+q)})\big)$$

[GN26, Definition 3.6.3]. It is alternating, being the alternation of a multilinear map, and its local representatives are finite sums of compositions of β with local representatives of ω and η, hence smooth.

Proposition 8.21. *The wedge product is bilinear; for* $p = q = 1$, $(\omega\wedge\eta)_x(v_1,v_2) = \beta(\omega_x v_1,\eta_x v_2) - \beta(\omega_x v_2,\eta_x v_1)$ *[GN26, Definition 3.6.3]; and* $\varphi^*(\omega\wedge\eta) = \varphi^*\omega\wedge\varphi^*\eta$ *[GN26, (3.28)].*

Proof. Direct from the definitions; for $p = q = 1$ the symmetric group has two elements. □

Proposition 8.22 (Products with functions)**.** *For a smooth function $f\colon M \to \mathbb{R}$, $(f\omega)_x = f(x)\,\omega_x$ defines a smooth form, and these products satisfy the module identities over $C^\infty(M,\mathbb{R})$. The form $f\omega$ is the wedge product of ω with the 0-form f for the scalar multiplication [GN26, Remark 3.6.5].*

Proof. Smoothness follows from the product rule for the scalar multiplication $\mathbb{R} \times F \to F$. For the wedge product with a 0-form, each permutation contributes $\operatorname{sgn}(\sigma)^2 f(x)\,\omega_x(v) = f(x)\,\omega_x(v)$, and there are $p!$ of them. □

Theorem 8.23 (Leibniz rule)**.** *For $\omega \in \Omega^p(M,E_1)$ and $\eta \in \Omega^q(M,E_2)$,*

$$\mathrm{d}(\omega \wedge_\beta \eta) = \mathrm{d}\omega \wedge_\beta \eta + (-1)^p\,\omega \wedge_\beta \mathrm{d}\eta.$$

(The degree $(p+1)+q$ of $\mathrm{d}\omega \wedge_\beta \eta$ is identified with $(p+q)+1$.)

Proof. The rule is local. By the product rule for β, the derivative of $z \mapsto \omega(z) \wedge_\beta \eta(z)$ at y in the direction u is $L(u) \wedge_\beta \eta(y) + \omega(y) \wedge_\beta L'(u)$, where $L(u)$, $L'(u)$ are the partial derivatives of the local representatives. By (3.33), it remains to prove two identities between alternating maps,

$$\sum_i (-1)^i (L(v_i) \wedge B)(\dots \hat{v}_i \dots) = (\mathrm{Alt}L) \wedge B, \qquad \sum_i (-1)^i (A \wedge L'(v_i))(\dots \hat{v}_i \dots) = (-1)^p A \wedge (\mathrm{Alt}L'),$$

where $(\mathrm{Alt}L)(v_0,\dots,v_p) = \sum_i (-1)^i L(v_i)(\dots \hat{v}_i \dots)$. Write $\mathrm{alt} f(v) = \sum_\sigma \operatorname{sgn}(\sigma) f(v \circ \sigma)$ for any function f of k vectors. Then $\mathrm{alt} f(v \circ \sigma) = \operatorname{sgn}(\sigma)\,\mathrm{alt} f(v)$, $\mathrm{alt} A = k!\,A$ for alternating A, and alternating a partial alternation $\sum_\tau \operatorname{sgn}(\tau) g(w \circ \tilde{\tau})$ over a subgroup of order $a!$ gives $a!\,\mathrm{alt} g$. The wedge product is $(p!\,q!)^{-1}\,\mathrm{alt}$ of $w \mapsto \beta(f(w_1,\dots,w_p), g(w_{p+1},\dots))$, and $\mathrm{Alt}L = (n!)^{-1}\mathrm{alt}(w \mapsto L(w_0)(w_1,\dots,w_n))$, because its i-th term is a reindexing by the cycle $(0\,1\cdots i)$, of sign $(-1)^i$. With these rules both sides of each identity become $(p!\,q!)^{-1}$ times the alternation of the same function, up to reindexing; in the second identity the reindexing is the cycle $(0\,1\cdots p)$, which gives the sign $(-1)^p$. □

8.6 Insertion and the Lie derivative

Definition 8.24 (Insertion)**.** For a smooth vector field X and $\omega \in \Omega^{n+1}(M,F)$, $(i_X\omega)_x(v_1,\dots,v_n) = \omega_x(X(x),v_1,\dots,v_n)$ [GN26, Definition 3.6.15]. In every chart, its local representative is $(y,v) \mapsto \omega(y)(\tilde{X}(y),v)$.

Definition 8.25 (Lie derivative)**.** We define $L_X = \mathrm{d} \circ i_X + i_X \circ \mathrm{d}$, with $i_X = 0$ on functions. Then L_X is a linear map of $\Omega^n(M,F)$, and Cartan's formula [GN26, Proposition 3.6.16(c)] holds by definition. Theorem 8.26 shows that this agrees with the definition [GN26, Definition 3.6.14].

Theorem 8.26 (The Lie derivative in charts)**.** *In every chart,*

$$(L_X\omega)(y)(v_1,\dots,v_n) = \partial_{\tilde{X}(y)}\omega(y)(v_1,\dots,v_n) + \sum_{k=1}^{n} \omega(y)(v_1,\dots,\mathrm{d}\tilde{X}(y)\,v_k,\dots,v_n).$$

Proof. Differentiating $z \mapsto \omega(z)(\tilde{X}(z),w)$ gives a partial derivative of ω and the term $\omega(y)(\mathrm{d}\tilde{X}(y)\,u, w)$. In $\mathrm{d}(i_X\omega) + i_X(\mathrm{d}\omega)$ the partial derivatives $\partial_{v_j}\omega(y)(\tilde{X}(y),\dots)$ cancel, and moving $\mathrm{d}\tilde{X}(y)\,v_j$ to the j-th place removes the signs. □

Theorem 8.27 (Evaluation on vector fields)**.** *For smooth vector fields $X_0, \dots, X_{n+1}$ and $\omega \in \Omega^{n+1}(M, F)$,*

$$\mathrm{d}\omega(X_0, \dots, X_{n+1}) = \sum_i (-1)^i X_i.\omega(\dots, \widehat{X_i}, \dots) + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots),$$

[GN26, Proposition 3.6.7, (3.29)], and $(L_Y\omega)(X_1, \dots, X_n) = Y.\omega(X_1, \dots, X_n) - \sum_j \omega(X_1, \dots, [Y, X_j], \dots, X_n)$ [GN26, (3.35)].

Proof. In a chart, $X_i.\omega(\dots, \widehat{X_i}, \dots)$ is a partial derivative of ω plus the terms with $\mathrm{d}\tilde{X}_k \cdot \tilde{X}_i$; the partial derivatives add up to $\mathrm{d}\omega$, and the other terms regroup in pairs into the brackets. The formula for L_Y follows in the same way from Theorem 8.26. □

Theorem 8.28. *For smooth vector fields X and Y [GN26, Proposition 3.6.16]:*

(a) $[L_X, L_Y] = L_{[X,Y]}$*;*

(b) $[L_X, i_Y] = i_{[X,Y]}$*;*

(d) $L_X \circ \mathrm{d} = \mathrm{d} \circ L_X$*;*

(e) L_X maps closed forms to exact forms.

Proof. (b) is a computation in a chart with Theorem 8.26: the difference is $\omega(y)(\mathrm{d}\tilde{Y}(y)\tilde{X}(y) - \mathrm{d}\tilde{X}(y)\tilde{Y}(y), \dots)$, and the first entry is the local representative of $[X, Y]$. (d) and (e) follow from Cartan's formula and $\mathrm{d} \circ \mathrm{d} = 0$, and (a) follows algebraically from (b), (d) and Cartan's formula. Glöckner and Neeb deduce (a)–(d) from the corresponding identities for Lie algebra cochains. □

Proposition 8.29 (Related vector fields)**.** *If X and Y are φ-related, then $\varphi^*(i_Y\omega) = i_X(\varphi^*\omega)$ and $\varphi^*(L_Y\omega) = L_X(\varphi^*\omega)$ [GN26, Exercise 3.6.6].*

Proof. The first identity holds because $T\varphi$ maps $X(x)$ to $Y(\varphi(x))$; the second follows from it, Cartan's formula and Theorem 8.17. □

8.7 De Rham cohomology

Definition 8.30 (De Rham cohomology)**.** Let $Z^n_{dR}(M, F) = \ker \mathrm{d}$ and $B^n_{dR}(M, F) = \mathrm{d}(\Omega^{n-1}(M, F))$, with $B^0_{dR}(M, F) = 0$. Then $B^n_{dR} \subseteq Z^n_{dR}$, and the de Rham cohomology is $H^n_{dR}(M, F) = Z^n_{dR}(M, F)/B^n_{dR}(M, F)$ [GN26, Definition 3.6.11].

Theorem 8.31. *A function is closed if and only if it is locally constant; hence $H^0_{dR}(M, F)$ is the space of locally constant functions $M \to F$ [GN26, Remark 3.6.12(b)].*

Proof. In a chart, a C^1 map with vanishing derivative is constant on convex open sets, and every point of a locally convex space has a convex open neighbourhood in a given open set. □

Proposition 8.32 (Functoriality)**.** *A smooth map $\varphi\colon N \to M$ induces linear maps $\varphi^*\colon H^n_{dR}(M, F) \to H^n_{dR}(N, F)$, $[\alpha] \mapsto [\varphi^*\alpha]$, with $\mathrm{id}^* = \mathrm{id}$ and $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$ [GN26, Section 3.6].*

Proof. By Theorem 8.17, pullbacks of closed (exact) forms are closed (exact); functoriality comes from Proposition 8.18. □

8.8 The Poincaré Lemma

Homotopies are smooth maps $H\colon \mathbb{R}\times M\to N$, with $H_t = H(t,\cdot)$; on $\mathbb{R}\times M$ we use the product charts $\mathrm{id}_{\mathbb{R}}\times\varphi$. Glöckner and Neeb use $[0,1]\times M$, a manifold with boundary; a homotopy on $[0,1]\times M$ can be precomposed with a smooth map $\mathbb{R}\to[0,1]$ which is 0 near $(-\infty,0]$ and 1 near $[1,\infty)$.

Definition 8.33 (Fiber integral)**.** Let F be Mackey complete. For $\omega\in\Omega^{k+1}(\mathbb{R}\times M,F)$, the fiber integral $I(\omega)\in\Omega^k(M,F)$ is

$$I(\omega)_x(v_1,\dots,v_k)=\int_0^1 \omega_{(t,x)}((1,0),(0,v_1),\dots,(0,v_k))\,dt$$

(weak integral; in a chart, the integrand is a smooth curve, whose weak integral exists because F is Mackey complete). In the chart φ, the local representative of $I(\omega)$ is the same integral of the local representative of ω in the chart $\mathrm{id}\times\varphi$, so $I(\omega)$ is smooth by the smoothness of parameter-dependent integrals.

Theorem 8.34 (Homotopy formula [GN26, (3.40)])**.** *With $i_t(x)=(t,x)$, $\mathrm{d}I(\omega)+I(\mathrm{d}\omega)=i_1^*\omega-i_0^*\omega$ for $\omega\in\Omega^{k+1}(\mathbb{R}\times M,F)$, and $I(\mathrm{d}f)=i_1^*f-i_0^*f$ for functions f.*

Proof. Glöckner and Neeb use Cartan's formula. We compute in a chart. Differentiating under the integral sign (which requires C^2 integrands in Mackey complete spaces), $\mathrm{d}I(\omega)(v_0,\dots,v_k)=\sum_j(-1)^j\int_0^1\partial_{(0,v_j)}\omega_{(t,x)}((1,0),(0,v_0),\dots,\widehat{(0,v_j)},\dots)\,dt$. In the formula (3.33) for $(\mathrm{d}\omega)_{(t,x)}((1,0),(0,v_0),\dots)$, the terms with $j\geq 1$ are the same with the opposite sign, and the remaining term is $\partial_t\omega_{(t,x)}((0,v_0),\dots)$. By the fundamental theorem of calculus, its integral is $\omega_{(1,x)}(\dots)-\omega_{(0,x)}(\dots)=(i_1^*\omega-i_0^*\omega)_x(v)$. □

Theorem 8.35 (Homotopy Lemma [GN26, Lemma 3.6.25])**.** *Let $H\colon\mathbb{R}\times M\to N$ be smooth and F Mackey complete. For a closed $\alpha\in\Omega^{k+1}(N,F)$, $H_1^*\alpha-H_0^*\alpha=\mathrm{d}I(H^*\alpha)$, and $H_1^*f=H_0^*f$ for closed functions f. Hence $H_0^*=H_1^*\colon H^n_{dR}(N,F)\to H^n_{dR}(M,F)$ for all n.*

Proof. Apply the homotopy formula to $\omega=H^*\alpha$, for which $\mathrm{d}\omega=H^*\mathrm{d}\alpha=0$, and use $i_t^*H^*=(H\circ i_t)^*=H_t^*$. □

Theorem 8.36 (Poincaré Lemma [GN26, Definition 3.6.27, Theorem 3.6.28])**.** *Let M be smoothly contractible: there are $p\in M$ and a smooth $H\colon\mathbb{R}\times M\to M$ with $H(0,x)=x$ and $H(1,x)=p$. Then every closed form of positive degree on M with values in a Mackey complete space is exact, i.e. $H^n_{dR}(M,F)=0$ for $n\geq 1$.*

Proof. $H_0=\mathrm{id}_M$ and H_1 is constant, so $H_1^*\alpha=0$ for $\deg\alpha\geq 1$. By the Homotopy Lemma, $-\alpha=\mathrm{d}I(H^*\alpha)$. □

Corollary 8.37 ([GN26, Corollary 3.6.29])**.** *Let $U\subseteq E$ be open and star-shaped with respect to $p\in U$. Every closed form of positive degree on U with values in a Mackey complete space is exact.*

Proof. Open subsets of manifolds are manifolds, because restrictions of smooth diffeomorphisms to open sets are smooth diffeomorphisms. With $c(t)=(1-t)^2/((1-t)^2+t^2)$, which is smooth with values in $[0,1]$, $c(0)=1$ and $c(1)=0$, the map $H(t,x)=p+c(t)(x-p)$ contracts U to p. □

Corollary 8.38 ([GN26, Corollary 3.6.26])**.** *Let G be a locally convex space and F a Mackey complete one. The projection $p\colon M\times G\to M$ induces isomorphisms $p^*\colon H^n_{dR}(M,F)\to H^n_{dR}(M\times G,F)$, with inverse i_0^*, where $i_0(m)=(m,0)$.*

Proof. $p \circ i_0 = \mathrm{id}_M$, and $H(t,(m,x)) = (m,tx)$ is a smooth homotopy from $i_0 \circ p$ to $\mathrm{id}_{M\times G}$, so $p^* i_0^* = (i_0 \circ p)^* = \mathrm{id}$ by the Homotopy Lemma. □

Theorem 8.39 (The Lie derivative along a local flow [GN26, Remark 3.6.17, (3.36)])**.** *Let X be a smooth vector field with a local C^2-flow* Fl*. Then* $\frac{d}{dt}\big|_{t=0}((\mathrm{Fl}_t)^*\omega)_x(v_1,\dots,v_n) = (L_X\omega)_x(v_1,\dots,v_n)$.

Proof. In a chart, differentiate $t \mapsto \omega(\varphi_t(y))(\mathrm{d}\varphi_t(y)v_1,\dots)$, where φ is the local flow: the base point moves with velocity $X(y)$, and $\partial_t \mathrm{d}\varphi_t(y)v = \mathrm{d}X(y)v$ at $t=0$ by exchanging the order of differentiation in $\partial_t\varphi = X\circ\varphi$. This is the local formula for L_X. Glöckner and Neeb assume C^1-flows; we do not claim this argument is new. □

Chapter 9

Vector bundles

Vector bundles with fibres modeled on locally convex spaces, defined by trivializing covers, pullbacks, Hom and dual bundles and the cotangent bundle, sections, connections on vector bundles with their curvature, induced connections and bundle metrics. As for connections on the tangent bundle (Chapter 10), bilinear families are encoded as maps of several variables, with joint smoothness.

9.1 Vector bundles and their connections

Definition 9.1 (Trivializing covers)**.** A *trivializing cover* of M, indexed by a set ι, is a family of open sets $U_i \subseteq M$, each contained in the domain of a chart φ_i of the atlas, together with a choice, for every $x \in M$, of an index $i(x)$ with $x \in U_{i(x)}$. The domains of the charts of the atlas form one.

Definition 9.2 (Vector bundles)**.** A *vector bundle* over M with fiber F, trivialized over a trivializing cover (U_i, φ_i), is given by coordinate changes $g_{ij}(x)\colon F \to F$ between the trivializations over U_i and U_j, linear for $x \in U_i \cap U_j$, satisfying the cocycle conditions, and such that $(y, \xi) \mapsto g_{ij}(\varphi_i^{-1} y)\xi$ is smooth on $\varphi_i(U_i \cap U_j) \times F$.

If F is not normable, $L(F)$ carries no vector topology for which the evaluation is continuous [Mai63, Satz 1], so one cannot ask $y \mapsto g_{ij}(y)$ to be a smooth map into $GL(F)$; joint smoothness of $(y, \xi) \mapsto g_{ij}(y)\xi$ is the substitute. For Banach spaces it is the classical condition only up to this difference of topologies on $L(F)$.

Theorem 9.3 (The total space is a manifold)**.** *The total space of a vector bundle with fiber F is a manifold modeled on $E \times F$, with transition maps $(y, \xi) \mapsto (\tau y, g(y)\xi)$. The tangent bundle is a vector bundle, with coordinate changes $(y, w) \mapsto \mathrm{d}\tau(y)w$.*

Proof. As for the tangent bundle: the transition maps are compositions of transition maps of M with the smooth maps $(y, \xi) \mapsto g(y)\xi$. □

Definition 9.4 (Pullback bundles)**.** Let $f\colon N \to M$ be smooth and V a vector bundle over M, trivialized over (U_i, φ_i). The *pullback* f^*V is the vector bundle over N trivialized over the sets $k.\mathrm{source} \cap f^{-1}(U_i)$, for k a chart of N, with coordinate changes $g_{ij} \circ f$. Its fiber over x is the fiber of V over $f(x)$.

The bundle f^*V need not be trivial over the domain of a chart of N, since f need not map it into a single U_i; this is why vector bundles are trivialized over general covers rather than over the atlas of the base.

Theorem 9.5 (Hom bundles, dual bundles and the cotangent bundle)**.** *Let V_1, V_2 be vector bundles over M with normed fibers F_1, F_2. There is a vector bundle $\operatorname{Hom}(V_1, V_2)$ with fiber $\mathcal{L}(F_1, F_2)$ (operator norm) and coordinate changes $A \mapsto g_2(x) \circ A \circ g_1(x)^{-1}$. In particular there is a dual bundle $V' = \operatorname{Hom}(V, \mathbb{R})$, and if E is normed, a cotangent bundle $T^*M = (TM)'$, with no completeness or reflexivity assumption.*

Proof. The cocycle conditions are those of V_1 and V_2. For smoothness, the maps $y \mapsto g_{ij}(y)$ are smooth for the operator norm, because $(y, \xi) \mapsto g_{ij}(y)\xi$ is smooth ([LCA], theorem on operator-valued maps, `contDiffMBOn_of_unc_clm`), and composition of operators is continuous and bilinear. □

If a fiber is not normable, $L(F_1, F_2)$ carries no vector topology for which the evaluation is continuous ([Mai63, Satz 1 and its footnote]), and these constructions are not available; Lempert notes the same restriction [Lem98, §2, p. 488].

Definition 9.6 (Connections on vector bundles)**.** A *connection* on a vector bundle is given by symbols $\Gamma_i(y)(a, \xi) \in F$ in the trivialization over each chart, smooth and bilinear, subject to the gauge transformation law

$$\Gamma_j(\tau y)(\mathrm{d}\tau(y)a, g(y)\xi) = g(y)\Gamma_i(y)(a, \xi) + (\partial_a g)(y)\xi,$$

where $\tau = j \circ i^{-1}$ and $g(y) = g_{ij}(i^{-1}y)$. In a trivialization, the covariant derivative of a section is $\nabla_X s = \mathrm{d}s \cdot X - \Gamma(X, s)$.

Lemma 9.7 (Calculus of the covariant derivative)**.** *Under the gauge transformation law, the covariant derivative transforms covariantly: if $X_j(\tau y) = \mathrm{d}\tau(y) X_i(y)$ and $s_j \circ \tau = g \cdot s_i$, then $\nabla_{X_j} s_j(\tau y) = g(y) \nabla_{X_i} s_i(y)$. For C^2 fields and sections, $\nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s = \Omega(X, Y)s$ pointwise.*

Proof. Product rules for $y \mapsto g(y)s(y)$ and $y \mapsto \Gamma(y)(Ay, By)$; in the curvature formula the terms with derivatives of the fields and the section cancel, using the symmetry of second derivatives of s, as in Lemma 10.10. □

Theorem 9.8 (The curvature is a tensor)**.** *The curvature forms of a connection on a vector bundle satisfy $\Omega_j(\tau y)(\mathrm{d}\tau\, u, \mathrm{d}\tau\, v)(g\xi) = g\, \Omega_i(y)(u, v)\xi$, and the second Bianchi identity holds in every trivialization. A connection on M is a connection on the tangent bundle; in particular its curvature is a tensor.*

Proof. Apply the curvature formula of Lemma 9.7 to the constant fields u, v and the constant section ξ in the chart i, and to their images $\mathrm{d}\tau(\sigma z)u$, $\mathrm{d}\tau(\sigma z)v$, $g(\sigma z)\xi$ in the chart j, where $\sigma = \tau^{-1}$. Covariant derivatives and brackets transform covariantly, so the left-hand sides correspond under $g(y)$. The argument uses neither partitions of unity nor bump functions, which need not exist on locally convex manifolds. □

Definition 9.9 (Sections and the covariant derivative)**.** A section s of a vector bundle has local representatives s_i with $s_j(\tau y) = g(y)s_i(y)$; those of a C^k section are C^k. For a connection, the covariant derivative $\nabla_X s$ of a section along a vector field is defined in the preferred chart at each point by $\mathrm{d}s_i \cdot X_i - \Gamma_i(X_i, s_i)$.

Theorem 9.10 (The global covariant derivative and curvature)**.** *For a C^1 section, $\nabla_X s$ is given by $\mathrm{d}s_j \cdot X_j - \Gamma_j(X_j, s_j)$ in every trivialization. For C^2 fields and sections, the curvature $R(X,Y)s = \nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s$ at x is the curvature form of the preferred chart evaluated at the values of X, Y, s at x.*

Proof. The first statement is Lemma 9.7 between the preferred chart and the chart j; the second is the local curvature formula, as for Theorem 10.11. □

Definition 9.11 (Bundle metrics and metric connections)**.** A *bundle metric* is a smooth family $h_i(y)$ of symmetric bilinear forms on F in each trivialization, with $h_j(\tau y)(g\xi, g\eta) = h_i(y)(\xi,\eta)$; positivity is not required. A connection is *metric* if $\partial_a h(\xi,\eta) = -h(\Gamma(a,\xi),\eta) - h(\xi,\Gamma(a,\eta))$ in every trivialization.

For a weak metric a metric connection need not exist; the statements below assume one.

Theorem 9.12 (Metric connections)**.** *For a metric connection, $Xh(s,t) = h(\nabla_X s, t) + h(s, \nabla_X t)$ for C^1 sections, and the curvature is skew-adjoint: $h(R(X,Y)s,t) + h(s,R(X,Y)t) = 0$.*

Proof. The first statement is the product rule for h. For the second, differentiate the compatibility condition in a second direction; the symmetry of the second derivatives of $y \mapsto h(y)(\xi,\eta)$ gives the identity, after the terms $h(\Gamma(a,\xi),\Gamma(b,\eta)) + h(\Gamma(b,\xi),\Gamma(a,\eta))$ cancel by symmetry. □

Theorem 9.13 (Pullback connections)**.** *Let $f\colon N \to M$ be smooth and ∇ a connection on V. In the trivialization of f^*V over $k.\mathrm{source} \cap f^{-1}(U_i)$, let $F = \varphi_i \circ f \circ k^{-1}$. The symbols $\Gamma_i(Fy)(\mathrm{d}F(y)a, \xi)$ define a connection $f^*\nabla$ on f^*V.*

Proof. The local representatives intertwine the coordinate changes, $F_q \circ \tau' = \tau \circ F_p$, so by the chain rule the gauge law of $f^*\nabla$ is the gauge law of ∇ at $F_p(y)$, in the direction $\mathrm{d}F_p(y)a$. □

Theorem 9.14 (Connections on Hom bundles)**.** *Let ∇^1, ∇^2 be connections on vector bundles V_1, V_2 with normed fibers. The symbols $\Gamma^{\mathrm{Hom}}(y)(a,A)\xi = \Gamma_2(y)(a,A\xi) - A\,\Gamma_1(y)(a,\xi)$ define a connection on $\mathrm{Hom}(V_1,V_2)$, the local form of $(\nabla_a A)\xi = \nabla^2_a(A\xi) - A(\nabla^1_a\xi)$.*

Proof. The map $y \mapsto \Gamma^{\mathrm{Hom}}(y)$ is smooth for the operator norm because $((y,a,A),\xi) \mapsto \Gamma^{\mathrm{Hom}}(y)(a,A)\xi$ is smooth ([LCA], theorem on operator-valued maps). For the gauge law, write the coordinate changes as $A \mapsto g_2 A g_1^{-1}$; differentiating $g_1(g_1^{-1}\xi) = \xi$ gives $\partial(g_1^{-1}) = -g_1^{-1}(\partial g_1)g_1^{-1}$, and the gauge laws of ∇^1 and ∇^2 give that of Γ^{Hom}. □

Corollary 9.15 (The dual connection)**.** *A connection on a bundle V with normed fiber induces a connection on the dual bundle V', with symbols $\Gamma'(y)(a,\alpha)\xi = -\alpha(\Gamma(y)(a,\xi))$.*

Proof. Apply Theorem 9.14 to V and the trivial bundle $M \times \mathbb{R}$ with the trivial connection, whose symbols vanish. □

Definition 9.16 (Refinements)**.** A trivializing cover (W_w) *refines* (V_k) along a map of indices r if $W_w \subseteq V_{r(w)}$. A vector bundle trivialized over (V_k) is trivialized over the refinement, with coordinate changes $g_{r(w)r(w')}$. Two covers (U_i), (V_k) have the common refinement $(U_i \cap V_k)$.

Theorem 9.17 (Connections on refinements; Hom bundles over different covers)**.** *A connection on V is written in the trivializations of a refinement with the symbols $\Gamma_{r(w)}(Fy)(\mathrm{d}F(y)a,\xi)$, where $F = \varphi_{r(w)} \circ \varphi_w^{-1}$. In particular, $\mathrm{Hom}(V_1,V_2)$ and its induced connection are defined for bundles trivialized over different covers, over their common refinement.*

Proof. As for pullback connections (Theorem 9.13), with the change of charts F in place of the local representative of f. □

Chapter 10

Connections

Connections on a manifold M modeled on a locally convex space E, given in charts by their Christoffel symbols: the covariant derivative of vector fields and of lifts of curves, the torsion and the curvature, connection forms and Cartan's structure equations, and geodesics as the curves with zero covariant acceleration. All the objects are defined chart by chart, subject to a change of charts law, and the main point in each case is that the resulting object does not depend on the chart. We follow Lang [Lan99] (§§IV.3, VIII.1–3, IX.1) and Chapter 6 of [Lar26].

The last section is closer to [Lar26]: a smooth field of projectors onto the tangent spaces of an embedded submanifold of a locally convex space defines a symmetric connection, whose covariant derivative is obtained by differentiating in the ambient space and projecting, and whose geodesics are the curves with normal acceleration.

There is no good topology on spaces of continuous linear operators, so bilinear families such as $x \mapsto \Gamma_x$ or $p \mapsto E_p$ are encoded as maps of several variables, linear in some of them, with joint smoothness. Existence of geodesics is only proved on Banach manifolds, in Chapter 11.

10.1 Connections and the covariant derivative

Let τ be a C^2 map between open subsets of locally convex spaces. We write $\mathrm{d}\tau(y)(a)$ for its derivative and $\mathrm{d}^{(2)}\tau(y;a,b)$ for the derivative of $y \mapsto \mathrm{d}\tau(y)(b)$ in the direction a; by the symmetry of second derivatives the latter is symmetric in (a,b). For two charts i, j of the atlas of M we write $\tau = j \circ i^{-1}$ for the transition map, defined on $i(U_i \cap U_j)$.

Definition 10.1 (Connection)**.** A *connection* on M assigns to each chart i of the atlas a family of maps $\Gamma_i(x) \colon E \times E \to E$, $x \in i(U_i)$, such that

1. each $\Gamma_i(x)$ is bilinear, and $(x,a,b) \mapsto \Gamma_i(x)(a,b)$ is smooth on $i(U_i) \times E \times E$;
2. (change of charts law) for every pair of charts i, j, every y in the domain of $\tau = j \circ i^{-1}$ and all $a, b \in E$,

$$\Gamma_j(\tau y)(\mathrm{d}\tau(y)(a), \mathrm{d}\tau(y)(b)) = \mathrm{d}\tau(y)(\Gamma_i(y)(a,b)) + \mathrm{d}^{(2)}\tau(y;a,b).$$

The maps Γ_i are the *Christoffel symbols* of the connection in the chart i.

The symbols are not tensorial: they transform affinely, the defect being the second derivative of the transition map. Our convention is that the first argument of Γ is the direction in which the base point moves. Continuity of each bilinear map $\Gamma_i(x)$ is a consequence of the joint smoothness, and no topology on the space of bilinear maps is needed.

Definition 10.2 (Local covariant derivative)**.** Let Γ be a family of bilinear maps on an open set $U \subseteq E$ and let $X, Y \colon U \to E$. The *local covariant derivative* is

$$(\nabla_X Y)(y) = \mathrm{d}Y(y)(X(y)) - \Gamma(y)(X(y), Y(y)).$$

Lemma 10.3 (Change of coordinates)**.** *Let τ be C^2, and let X_i, Y_i and X_j, Y_j be related by τ, that is $X_j \circ \tau = \mathrm{d}\tau \cdot X_i$ and $Y_j \circ \tau = \mathrm{d}\tau \cdot Y_i$, with Y_i, Y_j of class C^1. If Γ_i, Γ_j obey the change of charts law of Definition 10.1, then*

$$(\nabla_{X_j} Y_j)(\tau y) = \mathrm{d}\tau(y)((\nabla_{X_i} Y_i)(y)).$$

Proof. Differentiating $Y_j \circ \tau = \mathrm{d}\tau \cdot Y_i$ in the direction w gives, by the second-order chain rule, $\mathrm{d}Y_j(\tau y)(\mathrm{d}\tau(y)w) = \mathrm{d}^{(2)}\tau(y; w, Y_i y) + \mathrm{d}\tau(y)(\mathrm{d}Y_i(y)w)$. Take $w = X_i(y)$: the term $\mathrm{d}^{(2)}\tau(y; X_i y, Y_i y)$ cancels with the one in the change of charts law of Γ, and the rest is linear in $\mathrm{d}\tau(y)$. □

Definition 10.4 (Covariant derivative)**.** Let C be a connection on M and X, Y vector fields. The *covariant derivative* $\nabla_X Y$ is the vector field whose value at x is the local covariant derivative of the local representatives $\tilde{X}, \tilde{Y}$ of X, Y, computed in the preferred chart at x: $\mathrm{d}\tilde{Y} \cdot \tilde{X} - \Gamma(\tilde{X}, \tilde{Y})$.

Theorem 10.5 (Chart independence)**.** *Let Y be a C^1 vector field and X any vector field. In every chart j of the atlas, the local representative of $\nabla_X Y$ is $\mathrm{d}\tilde{Y}_j \cdot \tilde{X}_j - \Gamma_j(\tilde{X}_j, \tilde{Y}_j)$.*

Proof. Apply Lemma 10.3 to the transition map from the preferred chart to j, which is smooth; the local representatives of vector fields are related by its derivative. □

So the choice of the preferred chart in Definition 10.4 is only a matter of convenience. The usual algebraic rules follow at once from the local formula.

Proposition 10.6 (Rules of the covariant derivative)**.** *Let X, Y be vector fields with Y of class C^1, and let $\varphi \colon M \to \mathbb{R}$.*

1. $\nabla_{\varphi X} Y = \varphi \, \nabla_X Y$.
2. *If φ is C^1, then* $\nabla_X(\varphi Y) = X(\varphi)\, Y + \varphi \, \nabla_X Y$ *(Leibniz rule).*
3. *If X is C^k and Y is C^{k+1}, then $\nabla_X Y$ is C^k.*

Proof. The first two items are computed in the preferred chart: Γ is linear in its first argument, and the derivative of $\varphi\tilde{Y}$ in the direction $\tilde{X}$ is $\mathrm{d}\varphi(\tilde{X})\,\tilde{Y} + \varphi\,\mathrm{d}\tilde{Y}(\tilde{X})$. For the third, in each chart $y \mapsto \mathrm{d}\tilde{Y}(y)(\tilde{X}y)$ and $y \mapsto \Gamma(y)(\tilde{X}y, \tilde{Y}y)$ are compositions of C^k maps. □

Proposition 10.7 (Torsion)**.** *For vector fields X, Y, the vector $\nabla_X Y - \nabla_Y X - [X, Y]$ at x is, in the preferred chart, $\Gamma(\tilde{Y}, \tilde{X}) - \Gamma(\tilde{X}, \tilde{Y})$; in particular it only depends on $X(x)$ and $Y(x)$. We say that the connection is* symmetric *(torsion free) if its symbols are symmetric in every chart; in that case $\nabla_X Y - \nabla_Y X = [X, Y]$.*

Proof. The Lie bracket in a chart is $\mathrm{d}\tilde{Y} \cdot \tilde{X} - \mathrm{d}\tilde{X} \cdot \tilde{Y}$, and the derivative terms cancel. □

Remark 10.8. Symmetry of a connection is a chart-free condition: the change of charts law shows that if Γ_i is symmetric then so is Γ_j, because $\mathrm{d}^{(2)}\tau$ is symmetric and $\mathrm{d}\tau(y)$ is invertible on the overlap. In the Lean code the definition quantifies over all the charts of the atlas, which avoids this argument.

10.2 Curvature

We use Lang's sign for the curvature.

Definition 10.9 (Curvature)**.** For vector fields X, Y, Z, the *curvature* at x is

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Locally, for a family of symbols Γ on an open set of E, we write $\mathrm{d}\Gamma_x(v)(a,b)$ for the derivative of $x \mapsto \Gamma(x)(a,b)$ in the direction v, and

$$R_x(u,v)w = \mathrm{d}\Gamma_x(v)(u,w) - \mathrm{d}\Gamma_x(u)(v,w) + \Gamma_x(u,\Gamma_x(v,w)) - \Gamma_x(v,\Gamma_x(u,w)).$$

Lemma 10.10 (Local curvature formula)**.** *Let X, Y, Z be C^2 fields on an open set $U \subseteq E$ and Γ a C^1 family of bilinear maps. Then, pointwise on U, $\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = R(X,Y)Z$ with R the trilinear expression of Definition 10.9.*

Proof. Expand both covariant derivatives. The derivative of $y \mapsto \Gamma(y)(Ay, By)$ is $\mathrm{d}\Gamma_y(h)(Ay,By) + \Gamma_y(\mathrm{d}A(y)h, By) + \Gamma_y(Ay, \mathrm{d}B(y)h)$. All the terms containing second derivatives of Z cancel by the symmetry of second derivatives, and the terms containing first derivatives of X, Y, Z cancel with $\nabla_{[X,Y]}Z$ and among themselves. This is Lang [Lan99], Prop. IX.1.2. □

Theorem 10.11 (The curvature is tensorial)**.** *Let X, Y, Z be C^2 vector fields. Then $R(X,Y)Z$ at x is the trilinear expression $R_{\tilde{x}}(\tilde{X},\tilde{Y})\tilde{Z}$ in the preferred chart, evaluated at the values of X, Y, Z at x. In particular, it only depends on $X(x)$, $Y(x)$ and $Z(x)$.*

Proof. Near x the local representatives of $\nabla_Y Z$, $\nabla_X Z$ and $[X,Y]$ in the preferred chart at x are given by the local formulas (Theorem 10.5), and the local covariant derivative at a point only depends on the germ of the second field. Then apply Lemma 10.10. Compare [Lan99], Prop. IX.1.1–1.2. □

Proposition 10.12 (Symmetries of the curvature)**.** *For C^2 vector fields, $R(Y,X)Z = -R(X,Y)Z$. If the connection is symmetric, the first Bianchi identity holds: $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$.*

Proof. Both identities are checked on the trilinear expression. Skew-symmetry is immediate. For Bianchi, if Γ is symmetric then so is $\mathrm{d}\Gamma_x(v)$, and the twelve terms of the cyclic sum cancel in pairs. □

10.3 Connection forms and the structure equations

Classically, the connection form of a connection on a vector bundle is the $\mathrm{End}(F)$-valued 1-form $A(y)(v) = \Gamma(y)(v,\cdot)$ in a local trivialization, so that $\nabla = \mathrm{d} - A$, and the curvature is $\Omega = -\mathrm{d}A + A \wedge A$. If the fiber F is not normable, no vector topology on $L(F)$ makes the evaluation $L(F) \times F \to F$ continuous [Mai63, Satz 1], so A is not a differential form with values in a locally convex space of operators. We encode $\mathrm{End}(F)$-valued forms as families of F-valued forms depending linearly on an extra argument $\xi \in F$. The local statements are proved for an arbitrary fiber F; the tangent bundle is the case $F = E$.

Definition 10.13 (Connection and curvature forms)**.** Let $U \subseteq E$ be open and $\Gamma \colon U \times E \times F \to F$ smooth and bilinear in the last two variables. For $\xi \in F$, the *connection form* is the local 1-form $A^\xi(y)(v) = \Gamma(y)(v,\xi)$, and the *curvature form* is the local 2-form

$$\Omega^\xi(y)(u,v) = \partial_v \Gamma(y)(u,\xi) - \partial_u \Gamma(y)(v,\xi) + \Gamma(y)(u,\Gamma(y)(v,\xi)) - \Gamma(y)(v,\Gamma(y)(u,\xi)).$$

We write $(A \wedge A)(u,v)\xi = A(u)A(v)\xi - A(v)A(u)\xi$.

Theorem 10.14 (Cartan's structure equation)**.** $\Omega^\xi = -\mathrm{d}A^\xi + (A \wedge A)\xi$. *For the tangent bundle, Ω is the local curvature of Lemma 10.10, so the curvature of a connection on M, evaluated on vector fields, is computed in the preferred chart by $-\mathrm{d}A + A \wedge A$.*

Proof. $\mathrm{d}A^\xi(y)(u,v) = \partial_u \Gamma(y)(v,\xi) - \partial_v \Gamma(y)(u,\xi)$. □

Theorem 10.15 (Second Bianchi identity)**.** *The covariant exterior derivative of the curvature vanishes, $D\Omega = \mathrm{d}\Omega - [A \wedge \Omega] = 0$: for $u, v, t \in E$ and $\xi \in F$,*

$$\mathrm{d}\Omega(u,v,t)\xi = [A(u), \Omega(v,t)]\xi - [A(v), \Omega(u,t)]\xi + [A(t), \Omega(u,v)]\xi,$$

where $[A(a), \Omega(b,c)]\xi = A(a)\Omega(b,c)\xi - \Omega(b,c)A(a)\xi$.

Proof. By the structure equation and $\mathrm{d} \circ \mathrm{d} = 0$, $\mathrm{d}\Omega^\xi = \mathrm{d}((A \wedge A)\xi)$, which is computed with the product rule $\partial_c(\Gamma(a, \Gamma(b,\xi))) = \partial_c \Gamma(a, \Gamma(b,\xi)) + \Gamma(a, \partial_c \Gamma(b,\xi))$. The terms with three factors Γ on the right cancel by the Jacobi identity for commutators. □

Proposition 10.16 (Gauge transformations; torsion)**.** *Let $\tau = j \circ i^{-1}$ and $g(y) = \mathrm{d}\tau(y)$. The connection forms of a connection on M satisfy $A_j(\tau y)(gv)(gw) = g\, A_i(y)(v)w + (\partial_v g)(y)w$, so A is not a tensor. The torsion $T(u,v) = A(u)v - A(v)u = \Gamma(u,v) - \Gamma(v,u)$ is a tensor: $T_j(\tau y)(ga, gb) = g\, T_i(y)(a,b)$.*

Proof. The first statement is the change of charts law of the symbols. For the torsion, the inhomogeneous terms $\mathrm{d}^2\tau(a,b)$ and $\mathrm{d}^2\tau(b,a)$ cancel. □

Remark 10.17. For a vector bundle with fiber F the same formalism applies to the symbols in local trivializations, with transition functions $g(y) \in GL(F)$ in place of $\mathrm{d}\tau(y)$; for non-normable F one can only require $(y,\xi) \mapsto g(y)\xi$ to be smooth, not $y \mapsto g(y)$. Vector bundles, their connections and the tensorial transformation law of Ω are treated in Section 9.1.

10.4 Covariant derivative along curves and geodesics

We use the C^k curves of Definition 3.8 and the velocity of Definition 3.9. Let $\alpha \colon \mathbb{R} \to M$ be a curve. A *lift* of α is a map $\beta \colon \mathbb{R} \to TM$ with $\beta(t) \in T_{\alpha(t)}M$. In a chart i we write $\tilde\alpha = i \circ \alpha$ and $\mu_i(t)$ for the fiber coordinate of $\beta(t)$.

Definition 10.18 (Lifts of curves)**.** A lift β of α is C^k if, for every chart i, its fiber coordinate μ_i is C^k on $\alpha^{-1}(U_i)$.

Lemma 10.19 (Change of charts along a curve)**.** *For two charts i, j and $\tau = j \circ i^{-1}$, we have $\mu_j(t) = \mathrm{d}\tau(\tilde\alpha_i t)(\mu_i t)$ and $\dot{\tilde\alpha}_j = \mathrm{d}\tau(\tilde\alpha_i)(\dot{\tilde\alpha}_i)$, and*

$$\dot\mu_j = \mathrm{d}^{(2)}\tau(\tilde\alpha_i; \dot{\tilde\alpha}_i, \mu_i) + \mathrm{d}\tau(\tilde\alpha_i)(\dot\mu_i).$$

The fiber coordinate of the velocity in any chart j is $\dot{\tilde\alpha}_j$, and the velocity of a C^{k+1} curve is a C^k lift.

Proof. The first identity is the change of charts of the tangent bundle, and the others follow from the chain rule and the product rule applied to $t \mapsto \mathrm{d}\tau(\tilde\alpha_i t)(\mu_i t)$. □

Definition 10.20 (Covariant derivative along a curve)**.** Let C be a connection and β a lift of α. The *covariant derivative of β along α* at t is, in the preferred chart at $\alpha(t)$,

$$D_t\beta = \dot{\mu} - \Gamma(\tilde{\alpha})(\dot{\tilde{\alpha}}, \mu).$$

Theorem 10.21 (Chart independence along curves)**.** *Let α be a C^1 curve and β a C^1 lift. Read in any chart j around $\alpha(t)$, the covariant derivative $D_t\beta$ is $\dot{\mu}_j - \Gamma_j(\tilde{\alpha}_j)(\dot{\tilde{\alpha}}_j, \mu_j)$.*

Proof. By Lemma 10.19, $\dot{\mu}_j$ contains the term $\mathrm{d}^{(2)}\tau(\tilde{\alpha}_i; \dot{\tilde{\alpha}}_i, \mu_i)$, which cancels with the one in the change of charts law of Γ applied to $(\dot{\tilde{\alpha}}_i, \mu_i)$. The remaining terms are $\mathrm{d}\tau(\tilde{\alpha}_i)$ applied to the expression in the chart i. Compare [Lan99], §VIII.3. □

Definition 10.22 (Geodesic)**.** A curve α is a *geodesic* of the connection C if its velocity is parallel: $D_t\alpha' = 0$ for every t.

Theorem 10.23 (The geodesic equation)**.** *A C^2 curve α is a geodesic if and only if, in every chart j,*

$$\ddot{\tilde{\alpha}}_j = \Gamma_j(\tilde{\alpha}_j)(\dot{\tilde{\alpha}}_j, \dot{\tilde{\alpha}}_j).$$

Proof. By Theorem 10.21 and Lemma 10.19, in the chart j the covariant derivative of the velocity is $\ddot{\tilde{\alpha}}_j - \Gamma_j(\dot{\tilde{\alpha}}_j, \dot{\tilde{\alpha}}_j)$, transported by the invertible map $\mathrm{d}\tau$. For the converse it suffices to use the preferred chart. □

In terms of the spray of the connection (Proposition 11.2), the geodesic equation reads $\ddot{\tilde{\alpha}} = f(\tilde{\alpha}, \dot{\tilde{\alpha}})$ in every chart, which is the classical definition of the geodesics of a spray.

Remark 10.24 (Existence of geodesics). We do not claim that geodesics exist, nor that they are unique, for given initial position and velocity. In a Banach space the geodesic equation is an ordinary differential equation with a smooth right hand side, and the Picard–Lindelöf theorem gives local existence and uniqueness; this is how the exponential map is constructed in [Lan99] and [Lar26]. In a general locally convex space, and even in Fréchet spaces, ordinary differential equations with smooth right hand side may have no solutions, or many [Ham82, Part I, 5.6.1]. Therefore existence of geodesics, the exponential map and parallel transport along curves are not part of the theory developed here; they would have to be proved case by case, or under additional hypotheses on E. What we can do in general is to recognize geodesics, as in Theorem 10.33 below. When the model space is a Banach space, existence and uniqueness are proved in the next section.

10.5 Fields of projectors: derive and project

Let F be a locally convex space and $X \subseteq F$ an embedded submanifold modeled on a closed subspace $S \subseteq F$ (Definition 7.3). Recall that in a slice chart φ the tangent space at p is $T_pX = \{w \in F : \mathrm{d}\varphi(p)(w) \in S\}$, and that it does not depend on the slice chart (Definition 7.11). We write $\psi = \varphi^{-1}|_S$ for the local parametrization of X given by φ; then $\mathrm{d}\psi(y)$ maps S injectively onto $T_{\psi(y)}X$.

Definition 10.25 (Field of projectors)**.** A *field of projectors* onto the tangent spaces of X is a map $E\colon X \times F \to F$, $(p, w) \mapsto E_p w$, such that

1. each E_p is linear, $E_p^2 = E_p$, and the range of E_p is T_pX;
2. for every slice chart φ, the map $(y, w) \mapsto E_{\psi(y)}w$ is smooth on $\varphi(U) \cap S \times F$.

Remark 10.26. In [Lar26] the field of projectors is a smooth map $X \to \mathcal{B}(F)$ into the bounded operators of a Banach space. For a locally convex space F there is no good topology on the space of continuous operators, so we replace it by a map of two variables, linear in the second one, with joint smoothness. Note also that no complement of S in F is assumed: the kernels $\ker E_p$ are complements of the tangent spaces T_pX, but they are not required to come from a global splitting of F, nor to be related to each other.

Lemma 10.27 (Tangent vectors)**.** *$E_p w \in T_pX$ for every w, and $E_p w = w$ if $w \in T_pX$. In a slice chart, $\mathrm{d}\psi(y)(s) \in T_{\psi(y)}X$ for $s \in S$, hence $E_{\psi(y)}\mathrm{d}\psi(y)(s) = \mathrm{d}\psi(y)(s)$, and $\mathrm{d}\psi(y)$ is injective on S.*

Proof. The first two claims follow from $E_p^2 = E_p$ and the description of the range. Since $\mathrm{d}\varphi$ is a left and right inverse of $\mathrm{d}\varphi^{-1}$, we have $\mathrm{d}\varphi(\mathrm{d}\psi(y)s) = s \in S$, which gives the rest. □

Definition 10.28 (Symbols of a field of projectors)**.** In a slice chart φ, with $\psi = \varphi^{-1}|_S$ and $y \in \varphi(U) \cap S$, put

$$\Gamma_y(a,b) = -\,\mathrm{d}\varphi(\psi y)\Big(E_{\psi(y)}\big(\mathrm{d}^{(2)}\psi(y;a,b)\big)\Big), \qquad a,b \in S.$$

This vector lies in S, because $E_{\psi(y)}(\cdot)$ is tangent.

Equivalently, $\mathrm{d}\psi(y)(\Gamma_y(a,b)) = -E(\mathrm{d}^{(2)}\psi(y;a,b))$: the symbols are the tangent part of the second derivative of the parametrization, read in the chart.

Lemma 10.29 (Properties of the symbols)**.** *The symbols Γ_y are bilinear and symmetric, they are smooth on the chart domain, and for two slice charts they obey the change of charts law of Definition 10.1.*

Proof. Symmetry and bilinearity come from those of $\mathrm{d}^{(2)}\psi(y;\cdot,\cdot)$ (Schwarz), and smoothness from the smoothness of E in slice charts and of ψ. For the change of charts law let τ be the transition map between two slice charts, so that $\psi' \circ \tau = \psi$ near y. The second-order chain rule gives $\mathrm{d}^{(2)}\psi = \mathrm{d}^{(2)}\psi'(\mathrm{d}\tau\, a, \mathrm{d}\tau\, b) + \mathrm{d}\psi'(\mathrm{d}^{(2)}\tau(a,b))$. The last term is tangent, so E fixes it; applying $-\mathrm{d}\varphi' \circ E$ yields the law. □

Theorem 10.30 (The connection of a field of projectors)**.** *A field of projectors E onto the tangent spaces of X defines a symmetric connection on X, whose symbols in the chart induced by a slice chart φ are $\Gamma_y(a,b) = -\mathrm{d}\varphi(E(\mathrm{d}^{(2)}\psi(y;a,b)))$.*

Proof. The charts of the atlas of X are the restrictions of slice charts, and Lemma 10.29 provides all the axioms. This is the first part of [Lar26, Teorema 6.6.1]. □

The next result explains the title of the section: the covariant derivative is obtained by differentiating in F and projecting onto the tangent space.

Theorem 10.31 (Derive and project)**.** *Let A, B be fields on an open subset of the chart domain in S, with B of class C^1, and let $\widehat{B}(y) = \mathrm{d}\psi(y)(By)$ be the corresponding tangent field read in F. Then*

$$\mathrm{d}\psi(y)((\nabla_A B)(y)) = E_{\psi(y)}(\mathrm{d}\widehat{B}(y)(Ay)).$$

In other words, $\nabla_X Y = E(\mathrm{d}Y \cdot X)$ when the fields are read in the ambient space F.

Proof. By the second-order chain rule, $\mathrm{d}\widehat{B}(y)(Ay) = \mathrm{d}^{(2)}\psi(y;Ay,By) + \mathrm{d}\psi(y)(\mathrm{d}B(y)(Ay))$. Apply E: the second term is tangent and is fixed, and $E(\mathrm{d}^{(2)}\psi(y;Ay,By)) = -\mathrm{d}\psi(y)(\Gamma_y(Ay,By))$ by Definition 10.28. Compare [Lar26, §6.6]. □

Lemma 10.32 (Acceleration in a slice chart)**.** *Let c be a C^2 curve in the chart domain in S and $\alpha = \psi \circ c$. Then*

$$E_{\alpha(t)}(\ddot{\alpha}(t)) = \mathrm{d}\psi(ct)(\ddot{c}(t) - \Gamma_{c(t)}(\dot{c}(t), \dot{c}(t))),$$

where $\ddot{\alpha}$ is the acceleration of α in F. Moreover, a curve in X which is smooth as a curve in F is a C^k curve of X for every k.

Proof. We have $\dot{\alpha} = \mathrm{d}\psi(c)(\dot{c})$ and, by the chain rule and the product rule, $\ddot{\alpha} = \mathrm{d}^{(2)}\psi(c; \dot{c}, \dot{c}) + \mathrm{d}\psi(c)(\ddot{c})$; apply E as in the proof of Theorem 10.31. For the last claim, the local representative in a slice chart is the corestriction to S of $\varphi \circ \alpha$. □

Theorem 10.33 (Derive and project: geodesics)**.** *Let α be a C^2 curve in X. Then α is a geodesic of the connection defined by the field of projectors E if and only if its acceleration in F is normal:*

$$E_{\alpha(t)}(\ddot{\alpha}(t)) = 0 \qquad \text{for every } t.$$

Proof. By Theorem 10.23, α is a geodesic if and only if $\ddot{c} = \Gamma_c(\dot{c}, \dot{c})$ in every chart, where c is the local representative. By Lemma 10.32 this is equivalent to $E(\ddot{\alpha}) = 0$, since $\mathrm{d}\psi(c)$ is injective on S (Lemma 10.27). This is part of [Lar26, Teorema 6.6.1]. □

For instance, for the unit sphere of a (complete) Hilbert space with the orthogonal projectors onto $p^\perp$, the geodesics are the curves in the sphere with $\ddot{\alpha}(t)$ proportional to $\alpha(t)$. This example is not formalized.

Finally, we compare with the formulation of the book, where the connection is written in terms of the derivative of the field of projectors. We denote with $E_{*p,v}w$ the derivative of $q \mapsto E_q w$ at p in the direction v, computed in the slice chart.

Proposition 10.34 (The identity of the book)**.** *Let $v = \mathrm{d}\psi(y)(a)$ and $w = \mathrm{d}\psi(y)(b)$ be tangent vectors at $p = \psi(y)$, with $a, b \in S$. Then*

$$E_{*p,v}\, w = (1 - E_p)\, \mathrm{d}^{(2)}\psi(y; a, b).$$

In particular $E_{*p,v}\, w = E_{*p,w}\, v$.

Proof. Since $\mathrm{d}\psi(y')(b)$ is tangent, $E_{\psi(y')}\mathrm{d}\psi(y')(b) = \mathrm{d}\psi(y')(b)$ for y' near y. Differentiate this identity in the direction a: by linearity of E in the second variable, the left side gives $E_{*p,v}w + E_p\, \mathrm{d}^{(2)}\psi(y; a, b)$, and the right side gives $\mathrm{d}^{(2)}\psi(y; a, b)$. The symmetry follows from the symmetry of $\mathrm{d}^{(2)}\psi$. □

Remark 10.35. In [Lar26] the connection is presented through $\Gamma_p(v, w) = E_{*p,v}w$, which by Proposition 10.34 is the normal part $(1 - E)\mathrm{d}^{(2)}\psi(a, b)$ of the second derivative, read in F. The symbols of Theorem 10.30 are instead the tangent part, pulled back to S by the chart; the two descriptions are related through the decomposition $\mathrm{d}^{(2)}\psi = E\, \mathrm{d}^{(2)}\psi + (1-E)\mathrm{d}^{(2)}\psi$. In both cases the symmetry of the connection is a consequence of the symmetry of second derivatives of ψ. In the Lean code the connection is defined by the tangent part, and the book's formula enters only through Proposition 10.34.

Chapter 11

Sprays and geodesics

Sprays, the correspondence between quadratic sprays and symmetric connections, and geodesics on Banach manifolds: the geodesic equation in charts, local existence with uniform time, uniqueness, maximal geodesics, the exponential map and normal neighbourhoods, after Lang [Lan99, §§IV.3–4, VIII.5]. In general locally convex spaces there is no existence theorem for ordinary differential equations, so existence is only claimed for Banach models.

11.1 Sprays

A spray, or second order vector field, is a vector field ξ on TM such that $T\pi \circ \xi = \mathrm{id}$. In a chart it has the form $(x, v) \mapsto (x, v; v, f(x, v))$, and it is determined by its *local part* f. As we did for connections, we encode sprays through their local parts.

Definition 11.1 (Spray)**.** A *spray* on M assigns to each chart i of the atlas a smooth map $f_i \colon i(U_i) \times E \to E$ such that, for every pair of charts i, j, every y in the domain of $\tau = j \circ i^{-1}$ and every $v \in E$,

$$f_j(\tau y, \mathrm{d}\tau(y)(v)) = \mathrm{d}\tau(y)(f_i(y, v)) + \mathrm{d}^{(2)}\tau(y; v, v).$$

The spray is *quadratic* if $f_i(x, sv) = s^2 f_i(x, v)$ for all i, x, v and $s \in \mathbb{R}$.

The change of charts law is the change of charts law of the double tangent bundle $T(TM)$, restricted to the points of the form $(x, v; v, w)$. In [Lan99], §IV.3, a spray is by definition quadratic; we keep the two notions apart, since the quadratic condition is what is needed to pass to connections.

Proposition 11.2 (The spray of a connection)**.** *Every connection defines a quadratic spray, with local parts* $f_i(x, v) = \Gamma_i(x)(v, v)$.

Proof. Restrict the change of charts law of Γ to the diagonal $a = b = v$. Homogeneity of degree two follows from bilinearity. □

The converse requires recovering a bilinear map from its quadratic form, and checking that the polarization obeys the change of charts law of connections. For the first point, the following identity holds without any regularity assumption.

Lemma 11.3 (Homogeneous maps of degree two)**.** *Let* $g \colon E \to F$ *satisfy* $g(sv) = s^2 g(v)$ *for all* $s \in \mathbb{R}$, $v \in E$. *Then* $\mathrm{d}^{(2)} g(0; v, v) = 2\, g(v)$. *Moreover, if* B *is symmetric and bilinear, then* $B(a + b, a + b) - B(a - b, a - b) = 4B(a, b)$.

Proof. By homogeneity, $g(sv+tv)-g(sv) = ((s+t)^2-s^2)g(v)$, so the difference quotient can be computed exactly and $\mathrm{d}g(sv)(v) = 2s\,g(v)$ for every s. Differentiating once more along the line $s \mapsto sv$ at $s=0$ gives $2g(v)$. The polarization identity is immediate. □

Theorem 11.4 (The connection of a quadratic spray)**.** *Let f be a quadratic spray. In each chart i put*

$$\Gamma_i(x)(a,b) = \tfrac{1}{4}(f_i(x,a+b) - f_i(x,a-b)) = \tfrac{1}{2}\,\mathrm{d}^{(2)}(f_i(x,\cdot))(0;a,b).$$

Then the Γ_i are the symbols of a symmetric connection, whose spray is f.

Proof. By Lemma 11.3 applied to $g = f_i(x,\cdot)$ and to the symmetric bilinear map $\frac{1}{2}\mathrm{d}^{(2)}g(0;\cdot,\cdot)$, the two expressions agree; the second one is bilinear and symmetric, and $\Gamma_i(x)(v,v) = f_i(x,v)$. Smoothness of the symbols follows from the first expression. The change of charts law for Γ is obtained by polarizing the change of charts law of the spray, using that $\mathrm{d}\tau(y)$ is linear and $\mathrm{d}^{(2)}\tau(y;\cdot,\cdot)$ is symmetric and bilinear. This is [Lan99], Prop. IV.3.2 and §VIII.2. □

Thus quadratic sprays and symmetric connections are the same object, through $f(x,v) = \Gamma_x(v,v)$; in the rest of the chapter we work with connections.

11.2 Geodesics on Banach manifolds

In this section E is a Banach space. Then Michal–Bastiani and Fréchet derivatives agree, and in a chart i the geodesic equation is the second order equation

$$c'' = \Gamma_i(c)(c',c')$$

on the open set $\varphi_i(U_i) \subseteq E$, with Γ_i smooth and bilinear in the last two variables. We say that α is a *geodesic on an open set* $I \subseteq \mathbb{R}$ if it is continuous on I and satisfies this equation at every $t \in I$ in the preferred chart at $\alpha(t)$.

Theorem 11.5 (The geodesic equation does not depend on the chart)**.** *If α is continuous at t and satisfies the geodesic equation at t in a chart around $\alpha(t)$, it satisfies it in every chart around $\alpha(t)$.*

Proof. With $\tau = \varphi_j \circ \varphi_i^{-1}$ and $c = \varphi_i \circ \alpha$, $(\tau\circ c)'' = D^2\tau(c',c') + D\tau(c'')$, and the change of charts law of the symbols reads $\Gamma_j(\tau y)(D\tau\,a, D\tau\,b) = D\tau\,\Gamma_i(y)(a,b) + D^2\tau(a,b)$. □

Theorem 11.6 (Local existence of geodesics, with uniform time)**.** *Let y_0 be a point of the chart i. There is $r>0$ such that for every $K \geq 0$ there is $\varepsilon > 0$ with the following property: for every y with $\|y-y_0\| \leq r$ and every w with $\|w\| \leq K$ there is a geodesic α on $(-\varepsilon,\varepsilon)$, staying in the domain of i, with $\varphi_i(\alpha(0)) = y$ and $(\varphi_i\circ\alpha)'(0) = w$.*

Proof. Picard–Lindelöf for the first order system $(c,v)' = (v, \Gamma_i(c)(v,v))$, whose right hand side is C^1, hence locally Lipschitz; the time is uniform for small velocities, and the homogeneity of the equation ($t \mapsto c(st)$ is a solution with initial velocity $s\,c'(0)$) makes it uniform for velocities bounded by K. □

Theorem 11.7 (Uniqueness of geodesics)**.** *If M is Hausdorff, two geodesics on (a,b) with the same position and velocity at some $t_0 \in (a,b)$ agree on (a,b).*

Proof. The set of t near which the two geodesics agree is open. It is also closed in (a,b): at a limit point t the two curves meet, because M is Hausdorff; in a chart around this common point both solve the same equation, which has unique solutions since its right hand side is locally Lipschitz. By connectedness it is all of (a,b). □

Remark 11.8. The Hausdorff hypothesis cannot be dropped: on the line with two origins, the two copies of a line through the origins are geodesics of the flat connection with the same data at $t = 1$, and they differ at $t = 0$.

Theorem 11.9 (Extension of geodesics)**.** *Let M be Hausdorff, and let α be a geodesic on (a, b) such that $\alpha(t) \to q$ as $t \to b^-$ and the velocity of α, read in a chart around q, stays bounded as $t \to b^-$. Then α extends to a geodesic on (a, b') for some $b' > b$.*

Proof. Let ε be the uniform time of Theorem 11.6 around $\varphi_i(q)$. Take $t_1 > b-\varepsilon/2$ with $\alpha(t_1)$ close to q, and the geodesic μ with the position and velocity of α at t_1. By uniqueness $\mu(t-t_1) = \alpha(t)$ where both are defined, so the two curves glue to a geodesic on $(a, t_1 + \varepsilon)$, and $t_1 + \varepsilon > b$. □

Theorem 11.10 (Complete Finsler manifolds: geodesics extend [Lar26, Prop. 9.2.1])**.** *Let M be Hausdorff, with a Finsler length structure μ bounded below, such that M is complete for the symmetrized distance* $\max(d_b(x, y), d_b(y, x))$*. If α is a geodesic on (a, b) with $\mu(\dot\alpha)$ and $\mu(-\dot\alpha)$ bounded, then α extends to a geodesic on an interval $(a', b') \supsetneq (a, b)$ at both ends.*

Proof. Reparametrizing arcs of α by a smooth ramp gives C^1 competitors for d_b, so $d_b(\alpha(s), \alpha(t)) \le L(t-s)$ and the same backwards; thus $\alpha(t)$ is Cauchy as $t \to b^-$. Its limit in (M, d_b) is a limit in M because μ is bounded below, and the lower bound also bounds the velocity in a chart around the limit point. Theorem 11.9 applies; the end a is treated by reversing time. □

Remark 11.11. In [Lar26] the metric is invariant under the parallel transport of the spray, so geodesics have constant speed; we only use bounded speed. Since μ need not be reversible, both $\mu(\dot\alpha)$ and $\mu(-\dot\alpha)$ have to be bounded. With maximal geodesics, the statement "every geodesic is defined on $\mathbb{R}$" is Theorem 11.20.

Theorem 11.12 (Maximal geodesics [Lan99, §IV.3, §IV.4])**.** *Let M be Hausdorff and modeled on a Banach space, with a connection. The geodesics with initial velocity $v \in TM$ glue to a* maximal geodesic γ_v*, defined on an open interval $J_v \ni 0$: every geodesic with initial velocity v on an open interval $I \ni 0$ is the restriction of γ_v to I. For $s \neq 0$, $J_{sv} = s^{-1}J_v$ and $\gamma_{sv}(t) = \gamma_v(st)$; the maximal geodesic with zero initial velocity is constant and defined on $\mathbb{R}$.*

Proof. As for the maximal flow (Theorem 5.4), from local existence and uniqueness on intervals; homogeneity because $t \mapsto \alpha(st)$ is a geodesic with initial velocity $s\,\dot\alpha(0)$. The maximal geodesic is built directly from the geodesics, not as the projection of the maximal flow of the geodesic spray on TM; the two agree by Theorem 11.15. We do not claim this argument is new. □

Definition 11.13 (The exponential map [Lan99, §IV.4], [Lar26, §9.2])**.** The *exponential map* is $\exp v = \gamma_v(1)$, on $D(\exp) = \{v : 1 \in J_v\}$, and $\exp_p$ is its restriction to T_pM. Then $tv \in D(\exp)$ if and only if $t \in J_v$, and $\exp(tv) = \gamma_v(t)$; so the maximal geodesic is $t \mapsto \exp(tv)$.

Definition 11.14 (The geodesic spray [Lan99, §IV.3, pp. 97–98])**.** Let M be modeled on a Banach space, with a connection. The *geodesic spray* is the smooth vector field S on the manifold TM (modeled on $E \times E$) which, in the chart of TM over a chart i of M, reads

$$(x, w) \longmapsto (x, w;\ w, \Gamma_i(x)(w, w)).$$

It is well defined: under the transition maps of TTM, the local formulas in two charts agree by the change of charts law of the symbols. It is a second-order vector field, $\pi_* S(v) = v$.

Theorem 11.15 (Geodesics and the flow of the spray [Lan99, §IV.3, p. 97])**.** *The velocity $\dot\alpha$ of a geodesic is an integral curve of S; conversely, an integral curve β of S is the velocity of its projection $\pi\beta$, which is a geodesic. Hence, if M is Hausdorff, the maximal interval J_v of the geodesic with initial velocity v is the maximal interval of the integral curve of S through v, $\gamma_v(t) = \pi(\mathrm{Fl}_S(t, v))$ and $\mathrm{Fl}_S(t, v) = \dot\gamma_v(t)$.*

Proof. In the chart of TM over a chart i of M, an integral curve of S is a curve (x, w) with $x' = w$ and $w' = \Gamma_i(x)(w, w)$, which is the geodesic equation for x. □

Theorem 11.16 ($D(\exp)$ is open and exp is smooth [Lan99, §IV.4, p. 107], [Lar26, §6.1.2])**.** *Let M be Hausdorff and modeled on a Banach space, with a connection. The domain $D(\exp)$ is open in TM, contains the zero section and is star-shaped in the fibers, and $\exp\colon D(\exp) \to M$ is smooth. For each $p \in M$, $D(\exp_p)$ is an open star-shaped neighbourhood of 0 in T_pM, and $\exp_p$ is smooth on it.*

Proof. By Theorem 11.15, $D(\exp) = \{v : (1, v) \in D(\mathrm{Fl}_S)\}$ and $\exp = \pi \circ \mathrm{Fl}_S(1, \cdot)$; the domain of the maximal flow is open and the flow is smooth on it (Theorem 5.4, applied to the Hausdorff manifold TM). Star-shapedness is $tv \in D(\exp) \iff t \in J_v$. The library works with smooth manifolds, so exp is smooth, rather than C^{p-2} on a C^p manifold as in [Lan99]. That $\exp_p$ is a local diffeomorphism at 0 is Theorem 11.17. □

Theorem 11.17 ($\exp_p$ is a local diffeomorphism at 0 [Lan99, Thm. IV.4.1, p. 108], [Lar26, Obs. 6.2.2, p. 161])**.** *Let M be Hausdorff and modeled on a Banach space, with a connection, and let $p \in M$. Read in the preferred chart φ at p, the derivative at 0 of $w \mapsto \varphi(\exp_p w)$ is the identity: $d(\exp_p)(0_p) = \mathrm{id}$. Hence $\exp_p$ restricts to a diffeomorphism from an open neighbourhood of 0 in T_pM, contained in $D(\exp_p)$, onto an open neighbourhood of p.*

Proof. Since $\exp_p(tw) = \gamma_w(t)$ for $t \in J_w$ and $\gamma_w'(0) = w$, the derivative of $\varphi \circ \exp_p$ at 0 in the direction w is w. Michal–Bastiani smooth maps between open sets of Banach spaces are Fréchet smooth, so the inverse function theorem applies to $\varphi \circ \exp_p$; on the open set where its derivative is invertible the local inverse is smooth, and composing with φ^{-1} gives the diffeomorphism onto an open set of M. □

Theorem 11.18 ($G = (\pi, \exp)$ is a local diffeomorphism at 0_p [Lan99, Prop. VIII.5.1, p. 215], [Lar26, §6.2.1, p. 162])**.** *Under the hypotheses of Theorem 11.17, the map $G(v) = (\pi v, \exp v)$, $D(\exp) \to M \times M$, has differential $(a, b) \mapsto (a, a + b)$ at 0_p in charts, and restricts to a diffeomorphism from an open neighbourhood $V \subseteq D(\exp)$ of 0_p onto an open neighbourhood of (p, p). There is an open $W \ni p$ with $W \times W \subseteq G(V)$: any $x, y \in W$ are joined by a unique geodesic $t \mapsto \exp(tv)$, $0 \le t \le 1$, with $v \in V$, and v depends smoothly on (x, y) [Lar26, Teo. 6.2.3 (1)], [Lan99, Cor. VIII.5.2 (1)].*

Proof. In the chart of TM over φ and the chart $\varphi\times\varphi$ of $M\times M$, G reads $(x, u) \mapsto (x, \varphi(\exp(x, u)))$. Since $G(x, 0) = (x, x)$ and $d(\exp_p)(0) = \mathrm{id}$ (Theorem 11.17), its differential at $(\varphi p, 0)$ is $(a, b) \mapsto (a, a + b)$, invertible; the inverse function theorem concludes. Uniqueness is injectivity of G on V. □

Theorem 11.19 (Uniformly normal neighbourhoods [Lar26, Teo. 6.2.3 (2), p. 162], [Lan99, Cor. VIII.5.2 (2), p. 216])**.** *Under the hypotheses of Theorem 11.17, let φ be the preferred chart at p and read tangent vectors in the chart T of TM over φ. There are an open $W \ni p$ in the domain of φ and $r > 0$ such that for every $q \in W$ the vectors of T_qM with coordinates $(\varphi q, u)$, $\|u\| < r$, lie in $D(\exp)$, and $u \mapsto \exp_q(T^{-1}(\varphi q, u))$ is a diffeomorphism of $B_r(0) \subseteq E$ onto an open set containing W.*

Proof. The local inverse of G in charts (Theorem 11.18) is defined on a neighbourhood of $(\varphi p, 0)$ in $E \times E$, which contains a product of balls $B_r(\varphi p) \times B_r(0)$; its image is an open neighbourhood of $(\varphi p, \varphi p)$, which contains $B_\delta(\varphi p) \times B_\delta(\varphi p)$. Take $W = \varphi^{-1}(B_{\min(\delta,r)}(\varphi p))$. Since G preserves the first coordinate, slicing at $x = \varphi q$ gives the diffeomorphism at q. No local compactness is used: the radius r is uniform because it comes from the inverse function theorem applied to G at the single point 0_p. We do not claim this argument is new. The book also states the result for every $r \le R$ and that the joining geodesic stays in $\exp_x(B_R(0))$; these refinements are not formalized. □

Theorem 11.20 (Complete Finsler manifolds are geodesically complete [Lar26, Prop. 9.2.1])**.** *Under the hypotheses of Theorem 11.10, assume that every geodesic on a bounded open interval has bounded speed $\mu(\dot\alpha)$ and bounded backward speed $\mu(-\dot\alpha)$. Each assertion implies the next:*

1. *M is complete for the symmetrized distance;*
2. *every maximal geodesic is defined on $\mathbb{R}$;*
3. *M is geodesically complete: $D(\exp) = TM$;*
4. *there is $p \in M$ with $D(\exp_p) = T_pM$.*

Proof. $1 \Rightarrow 2$: if $b = \sup J_v$ were finite, Theorem 11.10 would extend γ_v past b, against maximality; boundedness below reduces to boundedness above through $J_{-v} = -J_v$. $2 \Rightarrow 3 \Rightarrow 4$ are immediate from Definition 11.13. We do not claim this argument is new. □

Remark 11.21 (What fails in infinite dimensions). In finite dimensions the Hopf–Rinow theorem makes 1–4 equivalent for Riemannian (and reversible Finsler) manifolds, and adds that $\exp_p$ is onto. In infinite dimensions the last conclusion fails: Atkin constructs a complete Riemannian Hilbert manifold, the unit sphere of $\mathbb{R} \times \ell^2$ with the metric induced by $|\xi|^2 + \langle Tx, x\rangle$ for a suitable strictly positive operator T, whose two poles are joined by no geodesic [Atk75, (3), p. 262, and Lemma 6, p. 265], so that $\exp_p$ is not onto (see also [Eke78, p. 288]). Even shortest curves need not exist: on the ellipsoid image of the unit sphere of a Hilbert space by $A = \operatorname{diag}(1, 1+2^{-1}, 1+2^{-2}, \dots)$, a complete Riemannian Hilbert manifold, e_0 and $-e_0$ are at distance π but every curve joining them is strictly longer (McAlpin, as presented in [Lar26, Ejemplo 10.4.2, pp. 301–302]). No local compactness is used in the proof above: the extension rests on the uniform existence time of geodesics with bounded initial velocity in a chart and on the lower bound of μ.

Chapter 12

Metric geometry

Metric spaces in which the distance is the infimum of lengths of curves, after Gromov, as in Chapter 8 of [Lar26]: the rectifiable distance, interior metric spaces, approximate midpoints, the metric Hopf–Rinow theorem, and their asymmetric counterparts. The lengths of curves are those of Mathlib, through the variation of a curve. Nothing in this chapter uses manifolds or the calculus; it is applied to Finsler manifolds in Chapter 13.

12.1 The rectifiable distance

Let (X, d) be a pseudo-metric space, with distances in $[0, \infty]$. The length of a continuous curve $\gamma\colon [a, b] \to X$ is the supremum of the sums $\sum_i d(\gamma(t_i), \gamma(t_{i+1}))$ over the partitions of $[a, b]$; in Mathlib this is the variation `eVariationOn`. Its basic properties are in Mathlib: refining a partition increases the sums, the length vanishes only for constant curves, it is additive under concatenation, invariant under monotone reparametrizations [Lar26, Obs. 8.1.1], lower semi-continuous under uniform convergence [Lar26, Prop. 8.1.3], and the arc length $t \mapsto \ell(\gamma|_{[a,t]})$ is continuous [Lar26, Lema 8.1.4].

Definition 12.1 (The rectifiable distance)**.** The rectifiable distance $d_\ell(x, y)$ is the infimum of the lengths of the continuous curves from x to y, and ∞ if there is none.

Proposition 12.2. *d_ℓ is a pseudo-distance, and $d \le d_\ell$. In particular the topology of d_ℓ is finer than that of d [Lar26, Obs. 8.1.7].*

Proof. Constant curves, reversed curves and concatenations give the three properties of a distance. The trivial partition gives $d(x, y) \le \ell(\gamma)$. □

Lemma 12.3 (When the topologies agree)**.** *The topologies of d and d_ℓ coincide if and only if, for every x and every $\varepsilon > 0$, the points close enough to x can be joined to x by curves of length $< \varepsilon$ [Lar26, Lema 8.1.8].*

Proof. This is the continuity of the identity $(X, d) \to (X, d_\ell)$, written with balls. □

Definition 12.4 (Interior metric spaces)**.** (X, d) is an interior metric space (a length space) if $d = d_\ell$ [Lar26, §8.1.1].

Proposition 12.5 (Approximate midpoints)**.** *In an interior metric space, for all x, y and $\varepsilon > 0$ there is z with $\max\{d(x, z), d(z, y)\} \le \frac{1}{2} d(x, y) + \varepsilon$ [Lar26, Problema 8.III].*

Proof. Take a curve of length $< d(x,y) + 2\varepsilon$. By the continuity of the arc length and the intermediate value theorem it has a point at half its length. □

Theorem 12.6 ([Lar26, Lema 8.1.10, Problemas 8.III–8.IV])**.** *An interior metric space satisfies the ball condition: for $r_1 + r_2 \le d(x,y)$, the closed balls $\bar{B}_{r_1}(x)$ and $\bar{B}_{r_2}(y)$ are at distance at most $d(x,y) - r_1 - r_2$. The ball condition is equivalent to having approximate midpoints, and a complete space with approximate midpoints is interior. So for complete spaces the three properties are equivalent.*

Proof. The converse direction builds dyadic chains of approximate midpoints with steps at most $(d(x,y)+\varepsilon)/2^n$, and the curve as the uniform limit of the corresponding step functions, which exists by completeness and is $(d(x,y)+\varepsilon)$-Lipschitz. This replaces the extension from dyadic rationals of [Lar26, Lema 8.1.9]. □

12.2 The metric Hopf–Rinow theorem

Lemma 12.7 (Approximate intermediate points)**.** *In an interior metric space, for $a, b \ge 0$ with $a + b = d(x,y)$ and $\varepsilon > 0$ there is z with $d(x,z) \le a + \varepsilon$ and $d(z,y) \le b + \varepsilon$.*

Proof. As in Proposition 12.5, with the point at arc length a. □

Lemma 12.8 (Lipschitz reparametrization)**.** *A curve of finite length ℓ can be reparametrized on $[0,1]$ as an ℓ-Lipschitz curve [Lar26, Obs. 8.1.5, Cor. 8.1.6].*

Proof. Compose with a right inverse of the normalized arc length; the curve is constant on the level sets of the arc length. □

Theorem 12.9 (Hopf–Rinow [Lar26, Teorema 8.2.3])**.** *Let X be a complete interior metric space.*

1. *If $\bar{B}_r(x)$ is compact for all $0 < r < \rho$, then $\bar{B}_\rho(x)$ is compact.*
2. *If X is locally compact, every closed ball is compact, and any two points are joined by a shortest curve, of length equal to their distance.*

Proof. (1) By Lemma 12.7 every point of $\bar{B}_\rho(x)$ is close to the compact ball $\bar{B}_{\rho-\delta}(x)$, so $\bar{B}_\rho(x)$ is totally bounded, and it is complete. (2) The supremum of the radii of compact balls cannot be finite: by (1) the ball of that radius is compact, and a finite cover by balls with compact closure, together with a Lebesgue number, gives a larger compact ball. For the shortest curves, reparametrize almost minimizing curves as in Lemma 12.8; they lie in a compact ball and are uniformly Lipschitz, and a cluster point in the product topology (Tychonoff) is a Lipschitz curve of length at most the distance. Here the formal proof departs from [Lar26], which uses Arzelà–Ascoli and the lower semicontinuity of the length. □

Definition 12.10 (Short geodesics, geodesic spaces)**.** $\gamma\colon [0,1] \to X$ is a short geodesic of length L if $d(\gamma(s),\gamma(t)) = L|s-t|$; its length is L. X is geodesic if it is interior and any two points are joined by a short geodesic [Lar26, Def. 8.1.11].

Proposition 12.11. *The sub-arcs of a curve realizing the distance realize the distance [Lar26, Obs. 8.1.13]; normalized by arc length such a curve is a short geodesic. Hence a complete, locally compact interior metric space is geodesic [Lar26, Teorema 8.2.3].*

Proof. Additivity of the length and the triangle inequality; for the normalization, the bounds $d(\eta(s),\eta(t)) \le L|s-t|$ and the triangle inequality through $\eta(s), \eta(t)$ give the reverse inequality. □

Remark 12.12. A space whose closed balls are compact is complete [Lar26, Obs. 8.2.4]. Being geodesic and locally compact does not imply completeness: the interval $(0, 1)$.

Example 12.13 (McAlpin: no shortest curves in infinite dimension). Let A be the diagonal operator on ℓ^2 with $a_0 = 1$, $a_n = 1 + 2^{-n}$, and $M = A(S)$, S the unit sphere, with lengths measured by the norm of ℓ^2. Then the distance in M between e_0 and $-e_0$ is π, but every piecewise C^1 curve in M joining them is strictly longer than π [Lar26, Ejemplo 10.4.2].

Proof. The curves $A(\cos(\pi t)e_0 + \sin(\pi t)e_n)$ have length at most $\pi a_n \to \pi$. Every curve on S from e_0 to $-e_0$ has length at least π, by the triangle inequality for angles. If $\beta = A\alpha$ is a curve in M, $\|\beta'\| = \|A\alpha'\| \geq \|\alpha'\|$, with equality only when $\alpha' \in \mathbb{R}e_0$, which forces α to stay at e_0. □

In finite dimension this cannot happen: a complete Riemannian manifold has shortest curves [Lar26, Cor. 10.4.3], and Corollary 13.55 gives the Finsler version.

12.3 Asymmetric distances

For a non-reversible Finsler structure d_b is not symmetric. Following Mennucci [Men13], an asymmetric distance b satisfies $b(x, x) = 0$ and the triangle inequality, and the topology is that of the symmetrized distance $\max\{b(x, y), b(y, x)\}$; forward and backward balls alone may generate different topologies [Men13, Example 4.1]. Lengths are forward lengths: suprema of $\sum b(\gamma(t_i), \gamma(t_{i+1}))$, which change under reversal of the curve [Men13, Remark 3.5].

Definition 12.14 (Asymmetric distances and forward lengths)**.** An asymmetric metric space carries b and the topology of its symmetrization. The forward length of a curve is its forward variation; b_ℓ is the infimum of forward lengths of continuous curves, and X is forward interior if $b_\ell = b$ (g-intrinsic in [Men13, Def. 3.16]).

Theorem 12.15 (Forward lengths)**.** *Forward lengths are additive and invariant under increasing reparametrizations; the running length of a continuous curve of finite forward length is continuous [Men13, Prop. 3.9]; $b \leq b_\ell$ and b_ℓ satisfies the triangle inequality. Forward interior spaces have approximate intermediate points, and curves of finite forward length admit forward Lipschitz reparametrizations on* $[0, 1]$.

Proof. The proofs of the symmetric case adapt, with care for the order of the arguments of b; the continuity of the running length follows [Men13, Appendix B.2]. □

Theorem 12.16 (Shortest forward curves)**.** *Let X be a Hausdorff forward interior space, $b(x, y) < \infty$, with compact closed forward ball $\{z : b(x, z) \leq b(x, y) + 1\}$ on which $b(w, z) \leq C\, b(z, w)$. Then there is a continuous curve from x to y of forward length $b(x, y)$.*

Proof. As in Theorem 12.9, with forward Lipschitz reparametrizations. The bounded asymmetry makes the limit curve Lipschitz for the symmetrized distance, hence continuous; without it the limit need not be continuous [Men13]. □

Theorem 12.17 (Non-reversible Finsler structures)**.** *For any Finsler length structure, $\ell^+(\gamma) \leq L_b(\gamma)$ and (M, d_b) is forward interior. A reversibility constant $\mu_x(-v) \leq \lambda\mu_x(v)$ gives $d_b(y, x) \leq \lambda d_b(x, y)$; if moreover μ is bounded below and the closed forward ball is compact, there are shortest forward curves.*

Proof. As Theorem 13.45, with forward sums; reversed chains give the bound on the asymmetry, and Lemma 13.50 makes (M, d_b) Hausdorff. □

Proposition 12.18 (Symmetrization)**.** *$\mu_s(v) = \mu(v) + \mu(-v)$ is a reversible Finsler length structure, and $\max\{d_b(x,y), d_b(y,x)\} \leq d_s(x,y)$. With a reversibility constant λ, $d_s \leq (1+\lambda)d_b$, and the identity is a homeomorphism between (M, d_b) and (M, d_s).*

Proof. Pointwise comparisons of the integrands pass to lengths of chains and to distances. □

So for the topology, bounded asymmetry reduces to the reversible case; lengths and shortest curves of d_b and d_s are, however, different, and without a reversibility constant the topology of d_s may be strictly finer.

Chapter 13

Metrics

This chapter collects the metric structures on a manifold M modeled on a locally convex space E. There are two families, and it is worth separating them from the start. A *Finsler* structure assigns to each tangent space a (possibly asymmetric) seminorm; it measures lengths of curves and gives a distance. A *bilinear* structure assigns to each tangent space a symmetric bilinear form, of any signature; it gives Riemannian, Lorentzian and pseudo-Riemannian metrics. The two families meet in exactly one place: a Riemannian metric is a Finsler metric, through $v \mapsto \sqrt{g_x(v,v)}$. Lorentzian and pseudo-Riemannian metrics are not.

In infinite dimensions each family splits further, according to two different questions. For Finsler structures the question is *how μ_p depends on p*: continuously on the tangent bundle, or only along curves. For bilinear structures the question is *which topology g_x induces on T_xM*: the one of the model (strong) or a coarser one (weak). We keep a separate word for each question, and we never use one for the other.

13.1 Finsler seminorms and two continuity conditions

Definition 13.1 (Finsler seminorm)**.** A *Finsler seminorm* on a topological vector space E is a continuous function $\mu\colon E \to [0,\infty)$ that is subadditive, $\mu(v+w) \le \mu(v) + \mu(w)$, and positively homogeneous, $\mu(cv) = c\,\mu(v)$ for $c \ge 0$. It is a *Finsler norm* if $\mu(v) = 0$ only for $v = 0$, and it is *reversible* if $\mu(-v) = \mu(v)$.

This is the notion of [Lar19, Def. 2.2]. A Finsler seminorm need not be symmetric: a reversible one is a continuous seminorm in the usual sense. Nothing is lost by allowing asymmetry at this stage.

Definition 13.2 (Field of Finsler seminorms)**.** A *field of Finsler seminorms* on M is a function $\mu\colon TM \to [0,\infty)$ whose restriction μ_x to each tangent space T_xM is a Finsler seminorm.

No condition relates μ_x and μ_y for $x \neq y$. There are two natural ones, and the whole point of this section is that they are different.

Definition 13.3 (Finsler metric)**.** A *Finsler metric* is a field of Finsler seminorms μ that is continuous as a function $\mu\colon TM \to \mathbb{R}$ on the tangent bundle.

Definition 13.4 (Finsler length structure)**.** A *Finsler length structure* is a field of Finsler seminorms μ such that $t \mapsto \mu(\dot\gamma(t))$ is continuous for every C^1 curve γ in M.

These are, respectively, the "continuous" Finsler metrics and the Finsler metrics "continuous along paths" of [Lar19, Def. 2.3]. We reserve the name *Finsler metric* for the first notion, since this is what the term means for any reader, and we give the second one its own name: a Finsler length structure is exactly what is needed to measure lengths of curves (compare the length structures of [BBI01, §2.1.1]), and nothing more.

Remark 13.5 (Two words we avoid). We do not speak of *weak* Finsler metrics: the word *weak* is used below for bilinear metrics, and there it refers to the topology of the fibers, not to the continuity on TM. We do not use *Finsler structure* alone either: Palais [Pal66, Def. 2.1 and 2.6] uses it for a much stronger notion, in which each μ_x is a norm inducing the topology of the fiber, locally uniformly equivalent to μ_{x_0}. That notion only makes sense for normable models and is not formalized here.

Theorem 13.6 (Velocities are continuous)**.** *The velocity $t \mapsto \dot\gamma(t)$ of a C^1 curve is a continuous curve in TM. Consequently, every Finsler metric is a Finsler length structure.*

Proof. In a chart i around $\gamma(t_0)$ the velocity reads $s \mapsto (i(\gamma(s)), (i\circ\gamma)'(s))$, which is continuous because $i\circ\gamma$ is C^1. The corresponding chart of TM is a homeomorphism onto its image, so the velocity is continuous near t_0. □

The converse fails, already on $\mathbb{R}^2$ (Section 19.2). The natural source of Finsler length structures that are not Finsler metrics are quotient metrics (Section 13.3).

13.2 Length and distance

The following notions only use the continuity of μ along curves. We state them for Finsler length structures, so that they apply to quotient metrics; for a Finsler metric they are obtained through Theorem 13.6.

Definition 13.7 (Length)**.** The length of a C^1 curve γ on $[a,b]$ is $L_\mu(\gamma) = \int_a^b \mu(\dot\gamma(t))\,dt$.

Proposition 13.8. *Lengths are nonnegative, additive on adjacent intervals, and constant curves have length zero.*

Proof. The integrand is continuous and nonnegative; additivity is that of the integral. The velocity of a constant curve vanishes and $\mu_x(0) = 0$ by positive homogeneity. □

Definition 13.9 (Chains of arcs)**.** A *chain of C^1 arcs* from x to y of length L is either empty ($x = y$, $L = 0$) or an arc $\gamma|_{[a,b]}$ of a C^1 curve with $\gamma(a) = x$, followed by a chain from $\gamma(b)$ to y; its length is the sum of the lengths of the arcs. Chains concatenate, and their lengths add.

A chain is a piecewise C^1 path, cut at its corners. Encoding the paths in this way makes concatenation a formality: there is no need to smooth the curve at the junctions, which on a manifold would require reparametrizations with vanishing velocity at the end points.

Definition 13.10 (Finsler distance)**.** The distance $d_\mu(x,y) \in [0,\infty]$ is the infimum of the lengths of the chains of C^1 arcs from x to y, and ∞ if there is none. It satisfies $d_\mu(x,x) = 0$.

This is [Lar19, Def. 2.7], where the infimum is taken over piecewise C^1 curves.

Theorem 13.11 (Triangle inequality)**.** *$d_\mu(x,z) \le d_\mu(x,y) + d_\mu(y,z)$ for all $x,y,z \in M$. Hence d_μ is an extended, possibly asymmetric, pseudo-distance.*

Proof. If there is no chain from x to y or from y to z, the right-hand side is ∞. Otherwise concatenate a chain from x to y with a chain from y to z, and take infima. □

Proposition 13.12 (Reversible fields give symmetric distances)**.** *The reversed curve $t \mapsto \gamma(-t)$ of a C^1 curve is C^1, with velocity $-\dot\gamma(-t)$. If μ is reversible, reversing an arc does not change its length, so chains can be run backwards and $d_\mu(x,y) = d_\mu(y,x)$.*

Proof. In a chart, the reversed curve is the composition of γ with $t \mapsto -t$, and the chain rule gives the velocity. The length follows by the change of variables $t \mapsto -t$. □

Remark 13.13 (Degenerate distances). The distance may vanish between distinct points even when every μ_x is a norm: Michor and Mumford proved that the geodesic distance of the L^2 metric on the diffeomorphism group of a manifold vanishes [MM05, Thm. 5.2]. So d_μ is in general only a pseudo-distance.

13.3 Lifting curves versus lifting neighborhoods

Let $\pi\colon N \to M$ be a C^1 map and μ a field of Finsler seminorms on M. Typically μ is defined through π, as a quotient metric $\mu(w) = \inf\{\mu_N(v) : T\pi\, v = w\}$, and what one controls is the function $\mu \circ T\pi$ on TN. The two continuity conditions then correspond to two lifting properties of π.

Lemma 13.14 (Chain rule for velocities)**.** *If π is C^1 and Γ is a C^1 curve in N, then $T\pi(\dot\Gamma(t)) = (\pi\circ\Gamma)^{\cdot}(t)$. The velocity at t only depends on the germ of the curve at t.*

Proof. In charts, this is the chain rule of Michal–Bastiani calculus [LCA] for the composition of the local representative of π with the local representative of Γ. □

Theorem 13.15 (Lifting curves gives a Finsler length structure)**.** *Suppose that every C^1 curve of M has, near each time, a C^1 lift to N, and that $\mu \circ T\pi$ is continuous along the velocities of C^1 curves of N. Then μ is a Finsler length structure.*

Proof. Near t_0, $\gamma = \pi\circ\Gamma$, so $\dot\gamma = T\pi(\dot\Gamma)$ by Lemma 13.14, and $\mu(\dot\gamma) = (\mu\circ T\pi)(\dot\Gamma)$ is continuous. □

Theorem 13.16 (Lifting neighborhoods gives a Finsler metric)**.** *If $T\pi\colon TN \to TM$ is a quotient map and $\mu \circ T\pi$ is continuous on TN, then μ is a Finsler metric.*

Proof. This is the universal property of quotient maps. □

Remark 13.17 (Submersions and homogeneous spaces). For Banach manifolds, a submersion (surjective differential with complemented kernel) has local smooth sections through every point, by the inverse function theorem. Then $T\pi$ is open, hence a quotient map, and C^1 curves lift locally (compose them with a local section): the hypotheses on π of both theorems hold. On locally convex manifolds there is no inverse function theorem and neither property is automatic; a curve may have C^1 lifts while $T\pi$ is not a quotient map. This is the situation of homogeneous spaces G/K of infinite-dimensional Lie groups in [Lar19, Lemma 3.19]: under a uniformity assumption on the metric of G, the quotient metric is continuous along paths that have piecewise smooth lifts, and continuous on the tangent bundle when π_* is a quotient map.

13.4 Bilinear metrics

Definition 13.18 (Smooth symmetric bilinear field)**.** A smooth field of symmetric bilinear forms on M assigns to each x a symmetric bilinear form g_x on T_xM, such that the local representatives $(y,a,b)\mapsto g_{i^{-1}(y)}(a,b)$ are smooth in every chart i. Each g_x is then jointly continuous, and $v\mapsto g_x(v,v)$ is continuous on TM. The field is *weakly nondegenerate* if $g_x(v,\cdot)=0$ forces $v=0$.

13.4.1 Riemannian metrics

Definition 13.19 (Weak Riemannian metric)**.** A *weak Riemannian metric* is a smooth field of positive definite symmetric bilinear forms. It is *strong* at x if the sets $\{v : g_x(v,v)<\varepsilon\}$ form a basis of neighborhoods of 0 in T_xM.

This is [Sch23, Definition 4.1]. *Weak* refers to the topology of the fibers: g_x is continuous on T_xM, but the norm $\sqrt{g_x(v,v)}$ may induce a strictly coarser topology than the one of the model. A strong Riemannian metric forces the model to be a pre-Hilbert space; it is a Hilbert space only if the model is complete, and we say so only in that case. On a Fréchet model that is not normable, such as the space C^∞_{per} of Chapter 18, every Riemannian metric is weak.

Theorem 13.20 (Riemannian metrics are Finsler metrics)**.** *If g is a weak Riemannian metric, $\mu_x(v)=\sqrt{g_x(v,v)}$ is a Finsler metric, and each μ_x is a reversible Finsler norm.*

Proof. Subadditivity is the triangle inequality, which follows from the Cauchy–Schwarz inequality $g_x(v,w)^2\le g_x(v,v)\,g_x(w,w)$; the latter is the nonpositivity of the discriminant of $t\mapsto g_x(v+tw,v+tw)\ge 0$. Continuity on TM is that of $v\mapsto g_x(v,v)$. □

So lengths and distances of Riemannian metrics are those of the previous sections.

13.4.2 Fundamental decompositions and the index

Definition 13.21 (Fundamental decomposition)**.** Let B be a symmetric bilinear form on a topological vector space E. A *fundamental decomposition* of B is a splitting $E=E_+\oplus E_-$ given by a continuous projection, B-orthogonal, with B positive definite on E_+ and negative definite on E_-.

Theorem 13.22 (The index is well defined)**.** *A negative definite subspace has (algebraic) dimension at most* $\dim E_-$*. In particular, two fundamental decompositions of B have negative parts of the same dimension.*

Proof. A negative definite subspace N meets E_+ only at 0, so the projection onto E_- along E_+ is injective on N. Apply this to the negative part of each decomposition. □

Remark 13.23 (Algebraic versus Hilbert dimension)*.* The argument works for any cardinal and needs no completeness. For a Krein space one usually measures E_- by its *Hilbert* dimension; for infinite index the two dimensions differ, and the invariance of the Hilbert dimension uses the completeness of (E_+,B) and $(E_-,-B)$ [Bog74, Corollary IV.7.4]. That version is formalized in lc-analysis (`KreinDecomposition.intrinsicDim_neg_eq` and `KreinDecomposition.intrinsicDim_neg_eq_card`).

Proposition 13.24. *A form with a fundamental decomposition is weakly nondegenerate.*

Proof. If $B(v,\cdot)=0$, write $v=a+b$ with $a\in E_+$, $b\in E_-$. By orthogonality $B(a,a)=B(v,a)=0$ and $B(b,b)=B(v,b)=0$, so $a=b=0$. □

13.4.3 Pseudo-Riemannian and Lorentzian metrics

Definition 13.25 (Weak pseudo-Riemannian metric)**.** A *weak pseudo-Riemannian metric* is a smooth field of symmetric bilinear forms admitting a fundamental decomposition of each tangent space. Its *index* at x is the dimension of the negative part of any such decomposition; it may be any cardinal.

Index 0 gives back weak Riemannian metrics; finite index corresponds to tangent spaces of Pontryagin type, infinite index to tangent spaces of Krein type. We keep the infinite index in the theory on purpose: Krein spaces appear naturally in operator theory.

Proposition 13.26. *A weak Riemannian metric is a weak pseudo-Riemannian metric of index* 0*.*

Proof. Take $E_+ = E$, $E_- = 0$ and the identity as projection. □

Definition 13.27 (Weak Lorentzian metric)**.** A *weak Lorentzian metric* is a weak pseudo-Riemannian metric of index 1 at every point.

Theorem 13.28 (Lorentzian metrics from timelike vectors)**.** *Let g be a smooth field of symmetric bilinear forms such that each T_xM contains a vector v with $g_x(v,v) < 0$ whose g_x-orthogonal is positive definite. Then $T_xM = v^\perp \oplus \mathbb{R}v$ is a fundamental decomposition, and g is a weak Lorentzian metric.*

Proof. The projection onto $v^\perp$ is $w \mapsto w - \frac{g_x(w,v)}{g_x(v,v)}\, v$, which is continuous because $g_x(v,\cdot)$ is. The remaining conditions are direct computations, and $\mathbb{R}v$ has dimension 1. □

So the decomposition of a Lorentzian tangent space is automatically topological: no closedness or complementation has to be assumed.

Remark 13.29 (Lorentzian metrics are not Finsler metrics)*.* Since $g_x(v,v)$ takes negative values, $\sqrt{g_x(v,v)}$ is not defined, and $\sqrt{|g_x(v,v)|}$ is not subadditive. The theory of lengths of a Lorentzian metric (timelike curves, the reverse triangle inequality) is different and is not developed here.

Remark 13.30 (What is not formalized)*.* We do not prove that the index is locally constant, nor that fundamental decompositions can be chosen continuously in x. These are genuine infinite-dimensional questions when the topology of g_x is weaker than that of the model. The Krein-space conditions (completeness of (E_+, g) and $(E_-, -g)$, and the topology of the majorant $g(P_+\cdot, P_+\cdot) - g(P_-\cdot, P_-\cdot)$) are not formalized either.

13.5 The Levi-Civita connection

In this section g is a smooth field of symmetric bilinear forms of any signature (Definition 13.18). In a chart i we write $G(y)(a,b)$ for its local representative and $\partial G(y)(u)(a,b)$ for the derivative of G in the base point, in the direction u. Recall that, with the conventions of Chapter 10, the covariant derivative is $\nabla_X Y = \mathrm{d}\tilde{Y} \cdot \tilde{X} - \Gamma(\tilde{X}, \tilde{Y})$ in a chart.

Definition 13.31 (Metric connections)**.** A connection is *metric* for g if in every chart

$$\partial G(y)(u)(a,b) = -G(y)(\Gamma_y(u,a), b) - G(y)(a, \Gamma_y(u,b)).$$

This is the chart expression of $X\, g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$: differentiate $y \mapsto G(y)(\tilde{Y}(y), \tilde{Z}(y))$ in the direction $\tilde{X}$ and use the bilinearity of $G(y)$.

Theorem 13.32 (Koszul formula)**.** *If the connection is symmetric and metric for g, then in every chart*

$$G(y)(\Gamma_y(u,a),b) = \tfrac{1}{2}(-\partial G(y)(u)(a,b) - \partial G(y)(a)(u,b) + \partial G(y)(b)(u,a)).$$

Proof. Write the compatibility condition for the three cyclic permutations of (u,a,b), add the first two and subtract the third; the symmetry of Γ_y and of $G(y)$ cancels four of the six terms. □

The proof is pure algebra at a point: it uses neither completeness, nor nondegeneracy, nor the sign of g.

Theorem 13.33 (Uniqueness of the Levi-Civita connection)**.** *If g is weakly nondegenerate, two symmetric metric connections have the same symbols in every chart, hence the same covariant derivative. In particular this holds for weak pseudo-Riemannian metrics of any index, finite or infinite, and for weak Riemannian and weak Lorentzian metrics.*

Proof. By Theorem 13.32, $G(y)(\Gamma_y(u,a) - \Gamma'_y(u,a),b) = 0$ for every b. The local representative $G(y)$ is weakly nondegenerate because g_x is and the coordinate changes are invertible. □

Theorem 13.34 (The metric spray)**.** *A symmetric connection is metric for g if and only if its spray $F_y(v) = \Gamma_y(v,v)$ satisfies, in every chart,*

$$G(y)(F_y(v),z) = -\partial G(y)(v)(v,z) + \tfrac{1}{2}\,\partial G(y)(z)(v,v).$$

Proof. On the diagonal $u = a = v$ the Koszul formula is the formula of the statement. Conversely, polarizing the formula of the statement in v gives back the Koszul formula, since Γ_y is symmetric and bilinear and $\partial G(y)$ is additive in each of its three arguments; and the Koszul formula gives back the compatibility with G. The additivity of $\partial G(y)(u)(a,b)$ in a and b follows by differentiating the identity $G(y+tu)(a+a',b) = G(y+tu)(a,b) + G(y+tu)(a',b)$, valid for small t. □

In the notation of [Lar26, Teorema 10.1.1], where $G(y)(a,b) = \langle a, g(y)b\rangle$ and $\partial G(y)(z)(a,b) = \langle a, g_{*y,z}\,b\rangle$, the formula of the statement is $\langle F_y(v), g(y)z\rangle = -\langle g_{*y,v}\,v, z\rangle + \frac{1}{2}\langle g_{*y,z}\,v, v\rangle$, and the Koszul formula is its polarization; see also [Lan99, Ch. VIII, Theorem 4.2, MS 1], whose bilinear map B_U is our Γ.

Proposition 13.35 (The obstruction to existence)**.** *Let B be a symmetric, weakly nondegenerate bilinear form and $D(u)(a,b)$ symmetric in (a,b). Symbols Γ that are symmetric and compatible with B exist if and only if every Koszul functional $b \mapsto \frac{1}{2}(-D(u)(a,b) - D(a)(u,b) + D(b)(u,a))$ is of the form $B(c,\cdot)$.*

Proof. One direction is Theorem 13.32. For the other, choose $\Gamma(u,a)$ representing the Koszul functional; it is symmetric by weak nondegeneracy, and compatible by the algebra of the previous proof. □

Remark 13.36 (Existence is not automatic). For a strong Riemannian metric on a Hilbert manifold every continuous functional is represented (Riesz), and the Levi-Civita connection exists [Lan99, Ch. VIII, Theorem 4.1]. For weak metrics, and every metric on a Fréchet manifold that is not normable is weak, the Koszul functional need not be represented: Schmeding gives a weak Riemannian metric on ℓ^2 with no metric spray [Sch23, Example 4.22]. Even when it is represented at every point, the symbols so obtained have to be smooth and satisfy the change of charts law; neither follows from the pointwise statement. Existence is therefore a property of the metric at hand, to be checked in examples.

The following example, after Schmeding, shows that the obstruction of Proposition 13.35 does occur.

Theorem 13.37 (A weak Riemannian metric without Levi-Civita connection)**.** *Let H be a real inner product space, T a continuous, symmetric and positive definite operator on H, and $v \in H$. Then $g_p(x,y) = e^{\langle v,p\rangle}\langle Tx, y\rangle$ is a weak Riemannian metric on H, and if $v \notin T(H)$ it has no symmetric metric connection. On ℓ^2, the operator $Tx = (x_n/(n+1))_n$ and the vector $v = (1/(n+1))_n$ satisfy these hypotheses.*

Proof. The derivative in the base point is $\partial G(p)(w)(x,y) = \langle v, w\rangle g_p(x,y)$. By Theorem 13.34, a symmetric metric connection would give $c \in H$ with

$$\langle Tc, b\rangle = -\langle v, x\rangle\langle Tx, b\rangle + \tfrac{1}{2}\langle Tx, x\rangle\langle v, b\rangle \qquad (b \in H),$$

that is, $T(c + \langle v, x\rangle x) = \frac{1}{2}\langle Tx, x\rangle v$. For $x = v \neq 0$ this puts v in $T(H)$. On ℓ^2, $Tc = v$ forces $c_n = 1$ for every n, and the constant sequence 1 is not in ℓ^2. □

When T is invertible, g is a strong metric and the Levi-Civita connection exists; the example needs a metric $\langle T\cdot, \cdot\rangle$ whose topology is strictly coarser than that of H. Schmeding's original example [Sch23, Example 4.22], the metric $e^{-\|p\|^2}\sum_n x_n y_n/n^3$ of [Sch23, Example 4.12] (after Magnani and Tiberio), uses the weight n^{-3} and the factor $e^{-\|p\|^2}$; the obstruction is the same.

13.6 Submanifolds: orthogonal projectors in any signature

Let β be a continuous symmetric bilinear form on F, of any signature: an inner product, or the indefinite form of a Krein or a Pontryagin space. Let $X \subseteq F$ be an embedded submanifold with a field of projectors P onto its tangent spaces (Section 10.5). In a slice chart, with $\psi = \varphi^{-1}|_S$, the *induced form* is $G(y)(a,b) = \beta(\mathrm{d}\psi_y a, \mathrm{d}\psi_y b)$. We say that P is *β-orthogonal* if $\beta(P_p w, t) = \beta(w, t)$ for every $w \in F$ and every $t \in T_pX$.

Lemma 13.38 (Orthogonal projectors need nondegenerate ranges)**.** *If β is weakly nondegenerate and Q is a β-orthogonal projector, no nonzero vector of the range of Q is β-orthogonal to the whole range.*

Proof. If $t \in \operatorname{ran} Q$ is orthogonal to $\operatorname{ran} Q$, then $\beta(w,t) = \beta(Qw,t) = 0$ for every $w \in F$, so $t = 0$. □

Theorem 13.39 (Gauss: the projector connection is metric)**.** *If P is β-orthogonal, the connection of P is metric for the induced form in every slice chart: $\partial G(y)(u)(a,b) = -G(y)(\Gamma(u,a),b) - G(y)(a,\Gamma(u,b))$.*

Proof. By the product rule, $\partial G(y)(u)(a,b) = \beta(\mathrm{d}\psi\, a, \mathrm{d}^2\psi(u,b)) + \beta(\mathrm{d}^2\psi(u,a), \mathrm{d}\psi\, b)$. On the other hand $\mathrm{d}\psi\,\Gamma(u,a) = -P(\mathrm{d}^2\psi(u,a))$, and since $\mathrm{d}\psi\, b$ is tangent, $\beta(P\mathrm{d}^2\psi(u,a), \mathrm{d}\psi\, b) = \beta(\mathrm{d}^2\psi(u,a), \mathrm{d}\psi\, b)$. The same holds with a and b exchanged. □

No sign condition on β is used: the computation is the one of the Riemannian case [Lar26, §10.1.1].

Theorem 13.40 (The projector connection is the Levi-Civita connection)**.** *If β is weakly nondegenerate and P is β-orthogonal, then in every slice chart the induced form is weakly nondegenerate, and every symmetric family of symbols compatible with it coincides with the symbols of P.*

Proof. By Lemma 13.38 the tangent spaces are nondegenerate, and $\mathrm{d}\psi_y$ maps S injectively onto T_pX; so $G(y)$ is weakly nondegenerate. Then Theorem 13.39 and the uniqueness given by the Koszul formula (Theorem 13.33) apply. □

Theorem 13.41 (The induced metric)**.** *The induced forms of the slice charts define a smooth field of symmetric bilinear forms on X, the* induced metric*, whose local representative in each slice chart is $\beta(\mathrm{d}\psi_y a, \mathrm{d}\psi_y b)$. If β is positive definite, it is a weak Riemannian metric.*

Proof. Define the form at x in the preferred slice chart. If τ is the change from a slice chart φ to the preferred chart φ', then $\psi' \circ \tau = \psi$ near y, and the chain rule gives $\mathrm{d}\psi'_{\tau y}(\mathrm{d}\tau_y v) = \mathrm{d}\psi_y v$. So the local representative in the chart of φ is $\beta(\mathrm{d}\psi_y a, \mathrm{d}\psi_y b)$, which is smooth by the product rule. Positivity follows from the injectivity of $\mathrm{d}\psi_y$. □

Theorem 13.42 (Levi-Civita connection of a submanifold, any signature)**.** *If P is β-orthogonal, the connection of P is metric for the induced metric. If moreover β is weakly nondegenerate, the induced metric is weakly nondegenerate, and every symmetric connection which is metric for it has the covariant derivative of P: the Levi-Civita connection of X is obtained by deriving in F and projecting.*

Proof. In each chart of the atlas of X the local representative is the induced form, so Theorem 13.39 gives the compatibility. Uniqueness is Theorem 13.33. □

Remark 13.43 (The indefinite case)*.* For an inner product on a Hilbert space every closed subspace has an orthogonal projector. For an indefinite β this fails, and Lemma 13.38 shows the first obstruction: a degenerate tangent space, such as that of a lightlike hypersurface of a Lorentzian space, admits no β-orthogonal projector, and the description of the Levi-Civita connection by "derive and project" is lost. In a Krein space a closed subspace has a β-orthogonal projector exactly when it is itself a Krein space with the induced form (Bognár, Theorem V.3.4, in lc-analysis); being closed, nondegenerate and decomposable is not enough [Bog74, Example V.3.3] (in lc-analysis).

The theory of ortho-complemented subspaces of Krein spaces used below (Bognár, Theorems V.3.1, V.3.4, V.3.5 and Example V.3.3), together with the majorant topology, the intrinsic dimension and continuous families of Krein spaces, is formalized in lc-analysis and documented in the chapter on Krein spaces of its blueprint.

Corollary 13.44 (Orthogonal fields of projectors in a Krein space)**.** *Let X be a submanifold of a real Hilbert space with $\beta(x, y) = \langle Jx, y\rangle$. A β-orthogonal field of projectors has ortho-complemented tangent spaces, which are Krein spaces, and continuous projectors. Conversely, a closed tangent space which is a Krein space has a continuous β-orthogonal projector; whether these can be chosen smoothly in p is not addressed.*

Proof. Bognár's Theorem V.3.4 [Bog74, Theorem V.3.4], formalized in lc-analysis (chapter on Krein spaces of its blueprint), items 2, 3 and 4. □

13.7 (M, d_b) is an interior metric space

Let μ be a reversible Finsler length structure on M (Definition 13.4). Its distance d_b (Definition 13.10) is then a pseudo-distance, and (M, d_b) is a pseudo-metric space; without reversibility d_b is only a quasi-distance and the symmetric theory of the previous section does not apply as such.

Theorem 13.45 ([Lar26, Prop. 9.1.2]). *C^1 curves are continuous for d_b, the metric length of a C^1 arc is at most its Finsler length, $\ell(\gamma) \le L_b(\gamma)$, and (M, d_b) is an interior metric space.*

Proof. $d_b(\gamma(s),\gamma(t)) \le |L_b(\gamma|_{[s,t]})|$ gives the continuity. For a partition, $\sum d_b(\gamma(t_i),\gamma(t_{i+1})) \le \sum L_b(\gamma|_{[t_i,t_{i+1}]}) \le L_b(\gamma)$. Hence d_ℓ is bounded by the length of every chain of C^1 arcs, so $d_\ell \le d_b$; the other inequality is Proposition 12.2. □

13.8 Bounded Finsler structures and the topology

In this section the model E is a normed space; from Lemma 13.49 on, it is complete and M is Hausdorff.

Definition 13.46 ([Lar26, Def. 9.1.3, Obs. 9.1.4]). μ is *bounded below* if every point has an open neighborhood U in the domain of a chart φ and $C > 0$ with $\|\varphi_{*x}v\| \le C\mu_x(v)$ for $x \in U$; it is *bounded above* if $m\mu_x(v) \le \|\varphi_{*x}v\|$ for some $m > 0$. A Finsler structure bounded above gives a *weak Finsler manifold*; bounded above and below, a *strong* one.

Remark 13.47 (Locally convex models). If the model is not normable, no Finsler norm μ_x can be bounded above and below by a norm inducing the topology of the model, since that topology would then be a norm topology. Strong Finsler manifolds are a phenomenon of normed models.

Lemma 13.48 (The integrated bound). *If μ is bounded below on U with constant C and E is complete, a C^1 curve that stays in U satisfies $\|\varphi(\gamma(t)) - \varphi(\gamma(s))\| \le CL_b(\gamma|_{[s,t]})$ [Lar26, (9.2)].*

Proof. Integrate the derivative of $\varphi \circ \gamma$ and bound its norm by $C\mu(\dot\gamma)$. □

Lemma 13.49 (Leaving a chart costs length). *Let μ be bounded below on U with constant C, and let $\bar B(\varphi(x_0), r)$ be the image of a part of U. A chain of C^1 arcs of length L starting at x_0 either has $CL \ge r$, or ends at a point $y \in U$ with $\|\varphi(y) - \varphi(x_0)\| \le CL$.*

Proof. For a single arc, let t_0 be the first time at which the arc leaves $U \cap \varphi^{-1}(B(\varphi(x_0), r))$. If $CL_b(\gamma|_{[a,t_0]}) < r$, the points $\varphi(\gamma(t))$, $t < t_0$, form a Cauchy family in a closed ball inside $\varphi(U)$; by completeness they converge, and since M is Hausdorff the limit is $\varphi(\gamma(t_0))$, which is then inside the ball: a contradiction. Chains follow by induction. □

Theorem 13.50 ([Lar26, Lema 9.1.8]). *If μ is bounded below, every point x_0 has a chart and $C, r > 0$ such that $d_b(x_0, y) < r/C$ implies that y is in the chart and $\|\varphi(y) - \varphi(x_0)\| \le Cd_b(x_0, y)$. Consequently d_b separates points, and, for a reversible μ, every open set of M is open for d_b.*

Proof. Apply Lemma 13.49 to chains whose length approaches $d_b(x_0, y)$. □

Theorem 13.51 (Lema 9.1.8 fails without completeness of the model). *For every incomplete normed space E_0 there is a Hausdorff manifold M modeled on $\mathbb{R} \times E_0$ with a reversible, non-degenerate Finsler metric, bounded above and below, such that the identity $(M, d_b) \to M$ is not continuous. This applies to c_{00} with the ℓ^2 norm. So the completeness of the model cannot be dropped from Theorem 13.50 nor from Corolario 9.1.9 of [Lar26].*

Proof. Every incomplete normed space carries a smooth curve $c\colon (-\infty, 1) \to E_0$ with $\|c'\| \le K$, $c'(s) \to 0$ as $s \to 1^-$ and no cluster point in E_0 as $s \to 1^-$: move along a non-convergent sequence with fast decaying increments by smooth steps on the intervals $[1 - 2^{-k}, 1 - 2^{-k-1}]$. Glue two copies of $E = \mathbb{R} \times E_0$ along $\{s < 1\}$ by $\tau(s, v) = (s, v + c(s))$. The result is Hausdorff because c has no cluster point. The norm of E read in the two charts is a Finsler metric, continuous

because $c' \to 0$ and bounded above and below because $\|c'\| \le K$. For $p = (1, 0)$ in the first copy and $q_\delta = (1, c(1-\delta))$ in the second, going down the first copy to $(1-\delta, 0)_1 = (1-\delta, c(1-\delta))_2$ and up the second gives $d_b(p, q_\delta) \le (2+K)\delta$, while no q_δ lies in the first copy, an open neighborhood of p. The exit argument of Lemma 13.49 fails exactly where completeness is used: the limit lies in the completion of E. We found no source for this counterexample; we do not claim it is new. □

Theorem 13.52 (Weak and strong Finsler manifolds)**.** *If μ is bounded above with constant m, then $d_b(x, y) \le \|\varphi(y) - \varphi(x)\|/m$ near each point [Lar26, (9.3)], so the topology of d_b is coarser than that of M. For a strong Finsler manifold (reversible, complete model, M Hausdorff) the identity is a homeomorphism between M and (M, d_b) [Lar26, Cor. 9.1.9].*

Proof. The segment from $\varphi(x)$ to $\varphi(y)$, reparametrized by a smooth transition function so that it is a C^1 curve on all of $\mathbb{R}$, has Finsler length at most $\|\varphi(y) - \varphi(x)\|/m$. The second statement combines this with Theorem 13.50. □

Lemma 13.53 ([Lar26, Lema 9.1.6])**.** *If μ, read in charts, is Lipschitz near the zero section, then μ is bounded above.*

Proof. Near $(\varphi(p), 0)$ the Lipschitz condition gives $\mu_q(w) \le K(\|\varphi(q) - \varphi(p)\| + \|\varphi_{*q} w\|)$; applied to a multiple of w of fixed small norm and combined with homogeneity, it gives $\mu_q(w) \le 2K\|\varphi_{*q} w\|$ for q close to p. □

Theorem 13.54 ([Lar26, Teorema 9.1.5])**.** *If the model is finite-dimensional, every Finsler metric whose seminorms are norms is strong.*

Proof. In a chart, $(y, v) \mapsto \mu(y, v)$ is continuous on the tangent bundle, hence attains a positive minimum and a maximum on the compact set $\bar{B} \times S$, closed ball times unit sphere. Homogeneity gives the bounds for every vector. □

Corollary 13.55 (Hopf–Rinow for Finsler manifolds [Lar26, Cor. 9.1.10])**.** *Let M be a Hausdorff manifold modeled on a finite-dimensional normed space, with a reversible Finsler metric whose seminorms are norms and with finite distance d_b. If (M, d_b) is complete, any two points are joined by a continuous curve whose length is their distance.*

Proof. By Theorem 13.54 the metric is strong; by Theorems 13.50 and 13.52, d_b is a distance and (M, d_b) is homeomorphic to M, hence locally compact. It is an interior space by Theorem 13.45, and Theorem 12.9 applies. □

Whether these shortest curves can be taken piecewise C^1 is a separate question [Lar26, Obs. 9.1.11].

Chapter 14

Local additions and manifolds of mappings

Glöckner and Neeb build the manifold structure of $C^\ell(M,N)$, for M compact and N with a local addition, in [GN26, §3.8, §4.1, §4.2]. This chapter follows them: local additions and the ways to produce them (open subsets, products, normalization, ambient formulas on submanifolds, real analytic local additions), the manifolds $C(K,N)$ and $C^\ell(M,N)$ with their exponential law, evaluation maps and tangent spaces, and the compact-open C^ℓ topology, described both through iterated tangent maps [GN26, Def. 4.1.2] and through jets on compact sets in charts. The models of $C^\ell(M,N)$ vary with the point, so these manifolds are atlases with varying models. As in the rest of the library, *no completeness* of the models is assumed.

14.1 Local additions

Definition 14.1 (Local additions [GN26, Def. 3.8.11])**.** A *local addition* on M is a smooth map $\Sigma\colon \Omega \to M$ on an open subset $\Omega \subseteq TM$ containing the zero section, with $\Sigma(0_x) = x$, such that $\theta = (\pi, \Sigma)\colon \Omega \to M \times M$ is a smooth diffeomorphism onto an open subset of $M \times M$. It is *normalized* if $T_{0_x}(\Sigma|_{T_xM}) = \mathrm{id}$ for all x; in the preferred chart φ_x at x, this reads $\mathrm{d}(\varphi_x \circ \Sigma(x,\cdot))(0)(w) = w$. In Lean the data is the map θ, an open partial homeomorphism $TM \to M \times M$ with source Ω, smooth with smooth inverse, whose first component is the bundle projection.

Proposition 14.2 (Open subsets of the model space)**.** *Let U be an open subset of E. Then $\Sigma(x,v) = x + v$ on $\Omega = \{(x,v) \in TU : x + v \in U\}$ is a normalized local addition on U, and $\theta(x,v) = (x, x+v)$ maps Ω onto $U \times U$.*

Proof. The charts of U are the inclusion, so $TU = U \times E$ with the identity as coordinate changes. In these coordinates θ and $\theta^{-1}(x,y) = (x, y-x)$ are restrictions of continuous linear maps. □

Proposition 14.3 (Normalization [GN26, Remark 3.8.16])**.** *Let Σ be a local addition on M. Then $\alpha_x = T_{0_x}(\Sigma|_{T_xM})$ is invertible for every x, the map $F\colon TM \to TM$, $F(v) = \alpha_{\pi v}^{-1} v$, is a diffeomorphism, and $\Sigma \circ F$ is a normalized local addition.*

Proof. In the preferred chart φ at x, the map $v \mapsto \varphi(\Sigma(x,v))$ has the local inverse $z \mapsto \theta^{-1}(x, \varphi^{-1} z)$ read in the tangent chart, so α_x is invertible by the chain rule. In an arbitrary chart, α_x and α_x^{-1} are conjugated by the derivative of the transition map and are partial derivatives of the chart representatives of θ and θ^{-1}; hence F and F^{-1} are smooth. Glöckner and Neeb

argue instead that θ restricts to diffeomorphisms of the fibres; we do not claim this argument is new. □

Lemma 14.4 (The tangent bundle of a product)**.** *The map $T(M_1 \times M_2) \to TM_1 \times TM_2$, $v \mapsto (T\mathrm{pr}_1(v), T\mathrm{pr}_2(v))$, is a smooth diffeomorphism.*

Proof. The coordinate changes of $T(M_1 \times M_2)$ between product charts are the products of the coordinate changes of TM_1 and TM_2. Hence, in charts, the inverse map $(v_1, v_2) \mapsto (v_1, v_2)$ is the regrouping $((a, b), (c, d)) \mapsto ((a, c), (b, d))$ of coordinates, which is continuous linear; the map itself is smooth by Theorem 3.5. □

Proposition 14.5 (Products of local additions [GN26, Lemma 3.8.17])**.** *If Σ_1 and Σ_2 are local additions on M_1 and M_2, then, identifying $T(M_1 \times M_2)$ with $TM_1 \times TM_2$, $\Sigma_1 \times \Sigma_2$ is a local addition on $M_1 \times M_2$.*

Proof. The map θ is $\theta_1 \times \theta_2$ composed with the identification of Lemma 14.4 and the regrouping $(M_1 \times M_1) \times (M_2 \times M_2) \cong (M_1 \times M_2) \times (M_1 \times M_2)$, all of which are smooth with smooth inverses. □

Proposition 14.6 (Local additions on embedded submanifolds from ambient formulas)**.** *Let $X \subseteq F$ be an embedded submanifold modeled on a closed subspace S. Let $\sigma \colon \Omega \to F$ and $\tau \colon V \to F$ be smooth on open subsets of $F \times F$ such that, for $p \in X$, $(p, 0) \in \Omega$, $\sigma(p, 0) = p$, and $w \mapsto \sigma(p, w)$ is a bijection from $\{w \in T_pX : (p, w) \in \Omega\}$ onto $\{q \in X : (p, q) \in V\}$ with inverse $q \mapsto \tau(p, q)$. Then $\Sigma(v) = \sigma(\pi v, v)$, with v read in F, is a local addition on X, normalized if $\mathrm{d}\sigma(p, \cdot)(0) = \mathrm{id}$ on T_pX.*

Proof. A map into X is smooth when it is smooth into F: in a slice chart φ its local representative is the corestriction to S of φ composed with the map into F. In the same chart, the fiber coordinate of a tangent vector v at p is $\mathrm{d}\varphi(p)w$, where $w \in T_pX \subseteq F$ is its ambient vector; so the ambient map $TX \to F \times F$, $v \mapsto (p, w)$, is smooth, and a map into TX is smooth when its composition with the ambient map is. Then $\theta = (\pi, \Sigma)$ and $\theta^{-1}(p, q) = (p, \tau(p, q))$ are smooth. Glöckner and Neeb obtain local additions from Lie group structures and from sprays [GN26, Prop. 3.8.12, Prop. 3.8.13]; we do not claim this argument is new. □

Proposition 14.7 (Injectivity near a compact set [GN26, Prop. 3.8.18, Cor. 3.8.19])**.** *A continuous map $f \colon X \to Y$ into a Hausdorff space, injective on a compact set K and locally injective at the points of K, is injective on an open neighborhood of K. Hence a local C^k-diffeomorphism ($k \in \mathbb{N}$ or $k = \infty$) into a Hausdorff manifold that is injective on a compact set K is a C^k-diffeomorphism from an open neighborhood of K onto an open set.*

Proof. The set $\{(x, y) : f(x) = f(y) \Rightarrow x = y\}$ is a neighborhood of each point of $K \times K$: of (x, x) by local injectivity, of (x, y) with $x \neq y$ because $f(x) \neq f(y)$ have disjoint neighborhoods. By the Wallace lemma it contains $A \times B$ for neighborhoods A, B of K, and f is injective on the interior of $A \cap B$. Glöckner and Neeb argue with nets; we do not claim this argument is new. For the second statement, the restriction is open and continuous, hence a homeomorphism onto its open image, and its inverse agrees near each point with a local inverse of f. The real analytic case is Corollary 14.11. □

Proposition 14.8 (Injectivity near a σ-compact set [GN26, Prop. 3.8.20, Cor. 3.8.21])**.** *Let $f \colon X \to Y$ be continuous, Y Hausdorff with every point having a compact neighborhood, and $A \subseteq X$ σ-compact such that $f(A)$ is closed, $f|_A$ is an embedding and f is locally injective at the points of A. Then f is injective on an open neighborhood of A. Hence a local C^k-diffeomorphism*

($k \in \mathbb{N}$ or $k = \infty$) into a Hausdorff, locally compact manifold (for instance a finite-dimensional one) which restricts to such an embedding on A is a C^k-diffeomorphism from an open neighborhood of A onto an open set.

Proof. Take a compact exhaustion $\emptyset = C_0 \subseteq C_1 \subseteq \cdots$ of a neighborhood of $f(A)$ with $C_n \subseteq \operatorname{int} C_{n+1}$. The "annuli" $O_j = \operatorname{int} C_{j+1} \setminus C_{j-1}$ meet only for $|i - j| \leq 1$. Let V_j be an open neighborhood of the compact set $A \cap f^{-1}(C_{j+1})$ on which f is injective (Proposition 14.7); then f is injective on $U = \bigcup_j f^{-1}(O_j) \cap V_j \cap V_{j+1}$. Glöckner and Neeb use their Lemma A.5.5 instead of the annuli, and the Hausdorff property of the domain, which is not needed here. The statement of [GN26, Cor. 3.8.21] concludes $K \subseteq U$, a misprint for $A \subseteq U$; its finite-dimensionality hypothesis is only used through local compactness of the target. We do not claim this argument is new. □

14.2 Real analytic local additions

Real analyticity is meant in Milnor's sense (Chapter 15): local restrictions of complex analytic maps on complexifications, with no completeness of the model.

Proposition 14.9 (Tangent bundles and products of real analytic manifolds)**.** *If M is a real analytic manifold modeled on E, its tangent bundle TM is a real analytic manifold modeled on $E \times E$. Products of real analytic manifolds are real analytic manifolds, and the projections and pairings of real analytic maps are real analytic.*

Proof. The transition maps of TM are $(y, w) \mapsto (\tau(y), \mathrm{d}\tau(y)\, w)$ for transition maps τ of M, and the differential of a real analytic map is real analytic: if g is a complex analytic extension of τ, then $\mathrm{d}g$ is one of $\mathrm{d}\tau$ (lc-analysis). Products of real analytic maps are real analytic because products of complex analytic extensions are complex analytic extensions. □

Definition 14.10 (Real analytic local additions [GN26, p. 325])**.** A local addition Σ on a real analytic manifold M is *real analytic* if $\theta = (\pi, \Sigma)$ is a real analytic diffeomorphism onto its open image. Then Σ is real analytic.

A real analytic manifold is a smooth manifold, so a local addition on it makes sense, and being real analytic is a property of a smooth local addition. The real analytic and the smooth maximal atlases differ; the charted space structure of TM is built from a real analytic atlas of M.

Corollary 14.11 (Real analytic local diffeomorphisms [GN26, Cor. 3.8.19, Cor. 3.8.21])**.** *Real analyticity of maps between real analytic manifolds is a local property. Hence Propositions 14.7 and 14.8 hold for real analytic local diffeomorphisms.*

Proof. Locality: real analyticity is checked in charts, and on chart domains it is local in the model [GN26, Lemma 2.2.3]. The rest is the topological part of the two propositions. □

Definition 14.12 (Split real analytic submanifolds [GN26, Def. 3.1.10])**.** Let $S \subseteq F$ be a complemented subspace, with a continuous linear projection $P \colon F \to S$. A subset $X \subseteq F$ is a *split real analytic submanifold* modeled on S if every point of X lies in a *real analytic slice chart*: a real analytic diffeomorphism φ between open subsets of F with $\varphi(U \cap X) = \varphi(U) \cap S$. The restrictions of these charts form an atlas of X. Such an X is an embedded submanifold.

Theorem 14.13 (Split real analytic submanifolds are real analytic manifolds)**.** *A split real analytic submanifold $X \subseteq F$ is a real analytic manifold modeled on S. A map into X is real*

analytic if and only if it is real analytic into F. The ambient map $TX \to F \times F$, $v \mapsto (\pi v, w)$, with $w \in T_pX \subseteq F$ the ambient vector of v, is real analytic, and a map into TX is real analytic as soon as its composition with the ambient map is.

Proof. As for smooth embedded submanifolds (Proposition 14.6), with real analytic maps in place of smooth ones. The only new point is corestriction: a map with values in S which is real analytic into F is real analytic into S because it equals its composition with P. □

Remark 14.14 (Why complemented). For smooth maps, a map with values in a closed subspace S is smooth into S. For real analytic maps in Milnor's sense this is less clear: a complex analytic extension to $D_{\mathbb{C}}$ of a map with values in S need not take values in $S_{\mathbb{C}}$. Assuming S complemented avoids the issue. In finite dimensions, and for closed subspaces of Hilbert spaces, this is no restriction; for general locally convex (even Banach) spaces it is. The real analytic atlas differs from the atlas of all smooth slice charts, so the two tangent bundles of X are different types in Lean, although the tangent spaces $T_pX \subseteq F$ agree.

Proposition 14.15 (Real analytic local additions from ambient formulas)**.** *Let $X \subseteq F$ be a split real analytic submanifold, and σ, τ real analytic maps on open subsets of $F \times F$ satisfying the hypotheses of Proposition 14.6. Then $\Sigma(v) = \sigma(\pi v, v)$ is a real analytic local addition on X.*

Proof. By Theorem 14.13, $\theta = (\pi, \Sigma)$ and $\theta^{-1}(p, q) = (p, \tau(p, q))$ are real analytic, because their compositions with the inclusion and with the ambient map are. Glöckner and Neeb do not discuss ambient formulas; we do not claim this argument is new. □

14.3 Manifolds of continuous maps

Let K be a compact Hausdorff space and N a manifold with a local addition $\Sigma\colon \Omega \to N$, $\theta = (\pi, \Sigma)$.

Definition 14.16 (The spaces Γ_f [GN26, 4.1.30, Lemma 4.1.31])**.** For $f \in C(K, N)$, $\Gamma_f = \{\tau \in C(K, TN) : \pi \circ \tau = f\}$, with the pointwise linear structure and the initial topology with respect to the fiber coordinates $\tau \mapsto (x \mapsto \tau(x)$ in the chart $i) \in C(L, E)$, for charts i of N and compact $L \subseteq K$ with $f(L)$ in the domain of i. It is a Hausdorff locally convex space, its topology is the one induced by $C(K, TN)$, and the fiber coordinates embed it as a closed subspace of a product.

Proposition 14.17 (Charts of $C(K, N)$ [GN26, proof of Thm. 4.2.2])**.** *The map $\varphi_f\colon O'_f \to O_f$, $g \mapsto \theta^{-1} \circ (f, g)$, from $O'_f = \{g : (f, g)(K) \subseteq \theta(\Omega)\}$ (open in the compact-open topology) onto $O_f = \Gamma_f \cap C(K, \Omega)$ (open in Γ_f) is a homeomorphism, with inverse $\tau \mapsto \Sigma \circ \tau$.*

Proof. Postcomposition with maps that are continuous on open sets is continuous in the compact-open topology (K is Hausdorff), and Γ_f carries the topology induced by $C(K, TN)$. □

Theorem 14.18 ($C(K, N)$ is a manifold [GN26, Thm. 4.2.2(a)–(c), $\ell = 0$])**.** *The charts φ_f, $f \in C(K, N)$, form an atlas of $C(K, N)$ with varying models Γ_f and smooth transition maps; the underlying topology is the compact-open topology, and the point evaluations $g \mapsto g(x)$ are smooth.*

Proof. A map into Γ_g is smooth if its fiber coordinates are, because these form a closed linear embedding into a product. The coordinates over a compact set are determined by those over the pieces of a finite cover on which f and g lie in charts i and j of N, again through a closed embedding. Over such a piece, the transition map $\varphi_g \circ \varphi_f^{-1}$ is the superposition operator $\gamma \mapsto h \circ \gamma$ of the smooth map $h(y, z, v) = \theta^{-1}(j^{-1}z, \Sigma(i^{-1}y, v))$ (in the chart j), applied to a curve that depends affinely on the section; superposition operators are smooth by [GN26, Prop. 1.7.1] (formalized in lc-analysis). Glöckner and Neeb deduce the smoothness from the exponential law for sections instead. □

14.4 Manifolds of C^ℓ maps

Let M be a compact smooth manifold, N a manifold with a local addition $\Sigma\colon \Omega \to N$, $\theta = (\pi, \Sigma)$, and $\ell \in \mathbb{N}_0 \cup \{\infty\}$.

Lemma 14.19 (Compact manifolds are finite-dimensional)**.** *If M is nonempty and compact, its model is locally compact, hence finite-dimensional by Riesz's theorem; so M is modeled on some $\mathbb{R}^n$, and only N may be infinite-dimensional. The results below use only the local compactness of the model, inside the proofs (the case $M = \emptyset$, where $C^\ell(M, N)$ is a point, is treated separately), so that their statements carry only the hypothesis that M is compact.*

Proof. A chart sends a compact neighborhood of a point onto a compact neighborhood in the model, and a topological group with a compact neighborhood is locally compact. □

Proposition 14.20 (Superposition operators on $C^r(U, E)$)**.** *Let Z be locally compact, $U \subseteq Z$ open, $r \in \mathbb{N}_0$, and $h\colon E \to F$ smooth on an open set O. If $g\colon X \to C^r(U, E)$ is C^k on an open set W and $g(w)(U) \subseteq O$ for $w \in W$, then $w \mapsto h \circ g(w) \in C^r(U, F)$ is C^k on W.*

Proof. Induction on r. Since $C^r(U, F)$ is a closed subspace of $\prod_{j \le r} C(U \times Z^j, F)$ [GN26, Lemma 1.7.15], it suffices to treat each $d^{(j)}$. For $j = 0$, restriction to the compact subsets of U (a closed embedding, Z being locally compact) reduces the claim to superposition operators on $C(C, E)$, C compact, which are smooth [GN26, Prop. 1.7.1]. For $j \ge 1$, $d^{(j)}(h\circ\gamma)$ depends continuously and linearly on $d(h\circ\gamma) = dh\circ(\gamma\circ\mathrm{pr}_1, d\gamma)$, a superposition by dh on $C^{r-1}(U\times Z, E\times E)$, and the induction hypothesis applies. Glöckner and Neeb obtain the smoothness they need from the exponential law [GN26, Prop. 4.1.34] instead. □

Lemma 14.21 (Restriction to open covers)**.** *For an open cover (V_a) of U, the restrictions $C^r(U, F) \to \prod_a C^r(V_a, F)$ form a linear topological embedding with closed range (compare [GN26, Lemma 1.7.43, Lemma 4.1.9]).*

Definition 14.22 (Jets on compact sets in charts; the topology of $C^\ell(M, N)$)**.** For charts j of M and i of N, a compact set K in the target of j (no interior is required) and $m \le \ell$, the *m-jet on K* of $g \in C^\ell(M, N)$ is $\mathrm{d}^{(m)}(i \circ g \circ j^{-1})|_{K\times H^m} \in C(K \times H^m, E)$, defined on the open set of maps with $g(j^{-1}(K))$ in the domain of i. The *compact-open C^ℓ topology* of $C^\ell(M, N)$, or *jet topology*, is the coarsest topology finer than the compact-open topology of $C(M, N)$ that makes all the jets continuous, with the compact-open topology on $C(K \times H^m, E)$. A map into $C^\ell(M, N)$ is continuous if and only if it is continuous into $C(M, N)$ and its jets are continuous where defined. For $\ell = \infty$, $C^\infty(M, N)$ carries the initial topology with respect to the inclusions into the $C^\ell(M, N)$ [GN26, Remark 4.1.3(b)]. This definition makes sense for every manifold M (compact or not) modeled on any Hausdorff locally convex space H; for Hausdorff M it is the topology of [GN26, Def. 4.1.2] (Theorem 14.41).

This is the weak (compact-open) C^r topology of [Hir76, Ch. 2, §1, pp. 34–35], with the uniform bounds on the derivatives over K replaced by the compact-open topology of $C(K \times H^m, E)$, as in the compact-open C^k topology of [GN26, Def. 1.7.10] for maps on open sets.

Definition 14.23 (The chart topology)**.** The *chart topology* of $C^\ell(M, N)$ is the coarsest topology finer than the compact-open topology for which the local representatives $g \mapsto i\circ g\circ j^{-1} \in C^\ell(V, E)$ are continuous on the open sets $\{g : g(j^{-1}(L)) \subseteq \operatorname{dom} i\}$, for charts j of M, i of N, compact L and open $V \subseteq L$; for $\ell = \infty$, the initial topology with respect to the finite orders. It is not the topology of $C^\ell(M, N)$: if the model of M is not locally compact, compact sets have empty interior, every such V is empty, and the chart topology collapses to the compact-open C^0 topology.

Proposition 14.24 (The chart topology and the jet topology)**.** *The jet topology is always finer than the chart topology. If the model of M is locally compact, in particular if M is compact, the two topologies coincide, $\ell \in \mathbb{N}_0 \cup \{\infty\}$.*

Proof. On a compact set $K \subseteq V$ the jets of the local representative on V are the jets of g on K, and the compact-open topology of $C(V \times H^m, E)$ is determined by the compact subsets of $V \times H^m$; this gives the first inclusion. Conversely, if the model is locally compact, a compact set K in the open domain of $i \circ g_0 \circ j^{-1}$ lies in the interior V of a compact set L in that domain; for g near g_0 the jets of g on K are restrictions of the differentials of the local representative on V. A nonempty compact manifold has a locally compact model (Lemma 14.19). □

Definition 14.25 (Γ_f [GN26, 4.1.30, Lemma 4.1.31])**.** The space Γ_f of C^ℓ sections of TN along f carries the initial topology with respect to the local representatives in charts; it is a Hausdorff locally convex space, embedded as a closed subspace of a product of spaces $C^\ell(D, E)$. For $\ell = \infty$ it carries the initial topology with respect to the finite orders [GN26, Remark 4.1.3(b)]. Glöckner and Neeb define this topology through $C^\ell(M, TN)$; the chart description used here corresponds to their Definition 4.1.16, Proposition 4.1.26 and the proof of Proposition 4.1.34.

Theorem 14.26 ($C^\ell(M, N)$ is a manifold [GN26, Thm. 4.2.2(a)–(c)])**.** *The charts $\varphi_f \colon g \mapsto \theta^{-1} \circ (f, g)$, $f \in C^\ell(M, N)$, are partial homeomorphisms for the compact-open C^ℓ topology onto open subsets of Γ_f, with inverses $\tau \mapsto \Sigma \circ \tau$; they form an atlas of $C^\ell(M, N)$ with varying models Γ_f and smooth transition maps, and the point evaluations are smooth.*

Proof. Near each point, f and g lie in charts i and j of N, and the local representative of $\varphi_g \circ \varphi_f^{-1}$ is the superposition operator of the smooth map $(y, z, v) \mapsto \theta^{-1}(j^{-1}z, \Sigma(i^{-1}y, v))$ applied to $(i \circ f, j \circ g, \tau)$, which depends affinely on τ (Proposition 14.20); Lemma 14.21 glues the local statements. The continuity of the charts is proved the same way, over compact neighborhoods in charts of M, in the chart topology, which is the topology of $C^\ell(M, N)$ for compact M (Proposition 14.24). The case $\ell = \infty$ follows from the finite orders. □

14.5 Smooth maps between atlases with varying models

Definition 14.27 (C^k maps for atlases with varying models [GN26, Def. 3.2.13])**.** A map $f \colon X \to Y$ between spaces with atlases with varying models is C^k on an open set U if it is continuous on U and all its local representatives $\psi_j \circ f \circ \varphi_i^{-1}$ are C^k on their (open) domains. A manifold modeled on a single space G is an atlas with varying models through all the charts of its atlas.

Proposition 14.28 (Basic properties)**.** *Being C^k can be checked in one pair of charts around each point; C^k maps compose, the identity is smooth, and restrictions of C^k maps to open subsets are C^k. For manifolds with a single model these are the C^k maps of Definition 2.5. Charts are smooth, with smooth inverses.*

Proof. Near each point, a local representative in arbitrary charts is the composition of the given one with transition maps, which are smooth. □

Proposition 14.29 (Equivalent atlases)**.** *Two atlases with varying models on the same space define the same smooth structure (the identity is smooth in both directions) if and only if all the transition maps between their charts are smooth, that is, if and only if their union is an atlas.*

Proof. The local representatives of the identity are the transition maps. □

Definition 14.30 (Tangent spaces and tangent maps)**.** The tangent space at x is the model of the preferred chart at x; the derivatives of the transition maps at the point are isomorphisms of topological vector spaces satisfying the cocycle condition. The tangent map $T_x f$ of a C^1 map is the derivative of its local representative in the preferred charts; in other charts it is conjugated by the changes of tangent coordinates. (In infinite dimensions this kinematic tangent space need not agree with the space of derivations.)

14.6 $C^{k,\ell}$ maps on products of manifolds

Proposition 14.31 (Local properties and the chain rule)**.** *Being $C^{k,\ell}$ on products of open sets is a local property. If f is $C^{k,\ell}$ on $U \times V$, α is C^k, β is C^ℓ and γ is smooth (with the obvious conditions on the images), and the second factor of the domain of β is locally compact, then $\gamma \circ f \circ (\alpha \times \beta)$ is $C^{k,\ell}$ (compare [GN26, Props. 1.7.26, 1.7.28]).*

Proof. By the exponential law [GN26, Thm. 1.7.30], $x \mapsto f(x, \cdot)$ is C^k into $C^\ell(V, F)$; pulling back by β is continuous linear [GN26, Prop. 1.7.11], superposition by γ is smooth (Proposition 14.20), and uncurrying needs local compactness. Glöckner and Neeb use Faà di Bruno's formula instead. □

Definition 14.32 ($C^{k,\ell}$ maps [GN26, Def. 3.5.24])**.** Let L carry an atlas with varying models (for instance a manifold) [GN26, Remark 3.5.28], and M, N be manifolds without boundary (manifolds with rough boundary are out of scope). A map $f\colon L \times M \to N$ is $C^{k,\ell}$ if around each point there are charts e, e', c and open sets $U \ni e(x)$, $V \ni e'(y)$ with $f(e^{-1}U \times e'^{-1}V) \subseteq \operatorname{dom} c$ and $c \circ f \circ (e^{-1} \times e'^{-1})$ of class $C^{k,\ell}$ on $U \times V$. Such maps are continuous, and, if the model of M is locally compact, their local representatives on all such boxes, in all charts, are $C^{k,\ell}$ [GN26, Remark 3.5.26(a)] (this holds when M is compact, Lemma 14.19).

14.7 The exponential law for manifolds of mappings

Theorem 14.33 (Exponential law [GN26, Thm. 4.2.2(a)])**.** *Let M be compact (without boundary) and N admit a local addition. For every space L with an atlas with varying models and every $k \in \mathbb{N}_0$, a map $g\colon L \to C^\ell(M, N)$ is C^k if and only if $g^\wedge\colon L \times M \to N$, $(x, y) \mapsto g(x)(y)$, is $C^{k,\ell}$ (for $\ell = \infty$: $C^{k,\ell'}$ for every $\ell' \in \mathbb{N}_0$).*

Proof. Locally, $g^\wedge$ reads $(a, b) \mapsto c(\Sigma(\tau(a)(b)))$ with $\tau = \varphi_f \circ g$; as a function of a this is a superposition operator applied to a C^k curve in a space of C^ℓ maps, and uncurrying gives $C^{k,\ell}$. Conversely, the local representatives of $\varphi_f \circ g$ are superposition operators applied to the curried local representatives of $g^\wedge$, over a finite cover of M by compact sets on which $g(x)$ stays in single charts of N. Glöckner and Neeb argue through the exponential law for sections [GN26, Prop. 4.1.34]. (Part (b) of [GN26, Thm. 4.2.2] holds by construction for the chart description of the compact-open C^ℓ topology; its agreement with [GN26, Def. 4.1.2] is Theorem 14.39.) □

Corollary 14.34 (The evaluation map [GN26, Thm. 4.2.2(c)])**.** *The evaluation map $\varepsilon\colon C^\ell(M, N) \times M \to N$ is $C^{k,\ell}$ for every k, and the point evaluations ε_x are smooth.*

Proof. Apply the exponential law to the identity; partial maps of $C^{k,\ell}$ maps are C^k. □

Corollary 14.35 (Uniqueness [GN26, Thm. 4.2.2(a)])**.** *An atlas with varying models on $C^\ell(M, N)$ satisfying the exponential law defines the same smooth structure as the atlas of the charts φ_f.*

Proof. Apply the exponential law to the identity in both directions: the evaluation map is $C^{k,\ell}$ for both atlases. □

Theorem 14.36 (Tangent spaces of $C^\ell(M,N)$ [GN26, Thm. 4.2.2(d)])**.** *For a local addition and $f \in C^\ell(M,N)$, the map $\Theta\colon T_fC^\ell(M,N) \to \Gamma_f$, $v \mapsto (T_f\varepsilon_x(v))_{x\in M}$, is an isomorphism of topological vector spaces. The smooth structure does not depend on the local addition.*

Proof. The chart φ_f sends f to the zero section, and in this chart ε_x reads $\tau \mapsto \Sigma(\tau(x))$; by the chain rule and the normalization (3.55), $T_f\varepsilon_x(v) = v(x)$, so Θ is the identity of Γ_f, the model of the chart at f, when the local addition is normalized. Glöckner and Neeb compute with geometric tangent vectors and assume the local addition normalized, which is no loss of generality (Proposition 14.3). For an arbitrary local addition Σ, the atlases of Σ and of its normalization are compatible, and Θ is the change of tangent coordinates between their charts at f; it is again $v \mapsto (T_f\varepsilon_x(v))_x$, a formula that involves no chart of $C^\ell(M,N)$. □

14.8 The compact-open C^ℓ topology via iterated tangent maps

Definition 14.37 (Iterated tangent bundles; [GN26, Def. 4.1.2])**.** The iterated tangent bundles T^jM and tangent maps T^jf are defined recursively. The compact-open C^ℓ topology on $C^\ell(M,N)$ is the initial topology with respect to $f \mapsto T^jf \in C(T^jM, T^jN)$, $j \le \ell$, compact-open. It is initial with respect to $f \mapsto f$ and $f \mapsto Tf \in C^{\ell-1}(TM,TN)$ (as in the proof of [GN26, Lemma 4.1.6]), and for $\ell = \infty$ it is the projective limit of the finite orders [GN26, Remark 4.1.3(b)].

Proposition 14.38 (Lemma 4.1.6 for the chart topology [GN26, Lemma 4.1.6])**.** *If the model of M is locally compact, the chart topology of $C^0(M,N)$ is the compact-open topology, and that of $C^{\ell+1}(M,N)$ is initial with respect to $f \mapsto f \in C(M,N)$ and $f \mapsto Tf \in C^\ell(TM,TN)$ (chart topology). (The chart topology is meant for locally compact models: otherwise it collapses to the C^0 topology.)*

Proof. In self tangent charts, $d(\operatorname{rep} f)$ is a component of a local representative of Tf; in general tangent charts the local representative of Tf is (F, dF) pulled back along a smooth map and composed with a smooth map. Localizations use compact neighborhoods. □

Theorem 14.39 (The chart topology and Definition 4.1.2 [GN26, Def. 4.1.2])**.** *If the model of M is locally compact, the chart topology of $C^\ell(M,N)$ is the compact-open C^ℓ topology of [GN26, Def. 4.1.2], for $\ell \in \mathbb{N}_0 \cup \{\infty\}$.*

Proof. Both topologies satisfy the recursion of [GN26, Lemma 4.1.6]; induction on ℓ, and the projective limit for $\ell = \infty$. □

14.9 The jet topology and Definition 4.1.2

The topology of $C^\ell(M,N)$ is the jet topology (Definition 14.22), which works for every Hausdorff locally convex model and every manifold M, compact or not. We compare it with the topology of [GN26, Def. 4.1.2].

Lemma 14.40 (Localizing compact sets)**.** *A space is* compact-localizable *if every compact set is covered by finitely many compact sets inside prescribed neighborhoods of its points. Hausdorff spaces and locally compact spaces are compact-localizable. Tangent bundles of Hausdorff manifolds*

are Hausdorff, so the iterated tangent bundles of a Hausdorff manifold, or of a manifold with a locally compact model, are compact-localizable.

Proof. For Hausdorff spaces this is the finite compact cover lemma used by Glöckner and Neeb [GN26, Lemma A.6.11]. □

Theorem 14.41 (The jet topology is the compact-open C^ℓ topology [GN26, Def. 4.1.2])**.** *The compact-open C^ℓ topology of [GN26, Def. 4.1.2] is always finer than the jet topology. For every Hausdorff manifold M, compact or not, modeled on any Hausdorff locally convex space, and for every M with a locally compact model, the two topologies coincide, $\ell \in \mathbb{N}_0 \cup \{\infty\}$. In particular, for M compact the topology underlying the atlases of $C^\ell(M,N)$ and $C^\infty(M,N)$ is that of [GN26, Def. 4.1.2] [GN26, Thm. 4.2.2(b)].*

Proof. Induction on the order through the tangent map, as in the proof of [GN26, Lemma 4.1.6]. In the tangent charts the local representative of Tg is $(y,w) \mapsto (F(y), \mathrm{d}F(y)\,w)$, $F = i \circ g \circ j^{-1}$, and the jets of $\mathrm{d}F$ and those of F determine each other by $\mathrm{d}^{(m+1)}F(x,v_0,v) = \mathrm{d}^{(m)}(\mathrm{d}F)((x,v_0),(v,0))$ [GN26, (1.36)] and the formula for $\mathrm{d}^{(m)}(\mathrm{d}F)$, the identities behind [GN26, Lemma 4.1.12]. Continuity of the jets of $\mathrm{d}F$ on the non-compact set $K \times H$ is reduced to compact subsets. Recovering the compact-open topology of $C(T^nM, T^nN)$ from compact sets inside charts needs Lemma 14.40; for non-Hausdorff manifolds with non-locally compact models only one inclusion is proved. We do not claim these arguments are new. □

Corollary 14.42 (The jet topology and the chart topology, via Definition 4.1.2)**.** *If the model of M is locally compact, the jet topology is the chart topology.*

Proof. This is Proposition 14.24; for a locally compact model it also follows from Theorems 14.41 and 14.39, both topologies being that of [GN26, Def. 4.1.2]. □

Theorem 14.43 (Targets in any universe)**.** *Theorems 14.39 and 14.41 hold for a target N in any universe; only M and the models share one universe. In particular, the topology of $C^\ell(M,N)$ is that of [GN26, Def. 4.1.2], $\ell \in \mathbb{N}_0 \cup \{\infty\}$, for every Hausdorff M [GN26, Thm. 4.2.2(b)], and so is the chart topology when the model of M is locally compact. Moreover, [GN26, Def. 4.1.2] is natural in the target: a diffeomorphism $\varphi\colon N \to N'$ induces a homeomorphism $f \mapsto \varphi \circ f$ between the topologies of [GN26, Def. 4.1.2] on $C^\ell(M,N)$ and $C^\ell(M,N')$.*

Proof. The iterated tangent bundles run over bundled manifolds whose carriers live in the universe $\max(u,v)$ of their tangent bundles, u that of the models. A target N in another universe is replaced by $\mathrm{ULift}\,N$, a manifold with the charts of N; $f \mapsto \mathrm{up} \circ f$ is a homeomorphism $C^\ell(M,N) \to C^\ell(M, \mathrm{ULift}\,N)$ for the jet topologies, the jets in corresponding charts being equal. Naturality follows from the chain rule $T^j(\varphi \circ f) = T^j\varphi \circ T^j f$, with $T^j\varphi$ a homeomorphism of inverse $T^j(\varphi^{-1})$; applied to $\mathrm{up}\colon N \to \mathrm{ULift}\,N$, it shows that the transported topology is that of [GN26, Def. 4.1.2] whenever N is itself a carrier. We do not claim this argument is new. □

Chapter 15

Analytic manifolds

Real and complex analytic manifolds modeled on locally convex spaces, following [GN26, Definitions 3.1.3–3.1.4 and 2.2.18]. Real analyticity is Milnor's notion: a map is real analytic if it extends to a complex analytic map on an open subset of the complexification $E_{\mathbb{C}} = E \times E$ (see the companion library `lc-analysis`). As for smooth manifolds, no completeness of the model space is assumed: composition, locality, "analytic implies smooth" and "complex analytic implies real analytic" hold for arbitrary Hausdorff locally convex spaces.

15.1 The analytic groupoids

Definition 15.1 (Analytic groupoids)**.** For a real locally convex space E, the *real analytic groupoid* consists of the homeomorphisms between open subsets of E which are real analytic, with real analytic inverse. For a complex locally convex space E, the *complex analytic groupoid* is defined in the same way with complex analytic maps.

Proof. These are structure groupoids: compositions of real (complex) analytic maps are real (complex) analytic [GN26, Proposition 2.2.4], and real (complex) analyticity is a local property [GN26, Lemma 2.2.3]. □

Proposition 15.2. *Every real analytic diffeomorphism is a smooth diffeomorphism, and (for a complex locally convex space) every complex analytic diffeomorphism is real analytic.*

Proof. Real analytic maps are smooth [GN26, Lemma 2.2.5]; complex analytic maps are real analytic [GN26, Proposition 2.2.11]. Neither statement uses completeness. □

Definition 15.3 (Analytic manifolds)**.** A *real (complex) analytic manifold* is a charted space whose transition maps belong to the real (complex) analytic groupoid. By Proposition 15.2, a complex analytic manifold is real analytic, and a real analytic manifold is smooth.

Remark 15.4. Both groupoids are closed under restriction to open sets, so open subsets of analytic manifolds (in particular of the model space) are analytic manifolds [GN26, Examples 3.1.8 and 3.1.9].

15.2 Analytic maps

Definition 15.5 (Analytic maps)**.** A map $f\colon M \to N$ is *real (complex) analytic* on an open set $U \subseteq M$ if it is continuous on U and its local representatives in the preferred charts are real (complex) analytic.

Lemma 15.6 (Insertion of charts)**.** *Near each point of the domain of $e' \circ f \circ e^{-1}$, this map factors as $(e' \circ c'^{-1}) \circ (c' \circ f \circ c^{-1}) \circ (c \circ e^{-1})$ with explicit open domains, for any other charts c, c' [GN26, Remark 3.1.5].*

Proof. Set theory. □

Theorem 15.7. *The local representatives of a real (complex) analytic map in arbitrary charts of the maximal atlases are real (complex) analytic. Compositions of analytic maps are analytic; identities and constants are analytic; real analytic maps are smooth; complex analytic maps are real analytic; and between open subsets of the model spaces, the notions agree with those of* `lc-analysis`*.*

Proof. Insert charts (Lemma 15.6) and use that the classes of analytic maps are closed under composition and local. □

Example 15.8 (Inversion in a continuous inverse algebra). The unit group $A^\times$ of a real continuous inverse algebra A is open in A, hence a real analytic manifold, and inversion is real analytic on it [GN26, Proposition 2.2.13].

15.3 Complexifications

Definition 15.9 (Complexification)**.** A complex analytic manifold M^* modeled on $E_{\mathbb{C}}$ is a *complexification* of a real analytic manifold M modeled on E if $M \subseteq M^*$ and each $x \in M$ lies in the domain U^* of a chart $\varphi\colon U^* \to U'$ of M^* with $\varphi(M \cap U^*) = E \cap U'$ such that $\varphi|_{M \cap U^*}$ is a chart of M [GN26, Definition 2.2.18]. In Lean the inclusion is an injective map $\iota\colon M \to M^*$. The space $E_{\mathbb{C}}$ is a complexification of E.

Lemma 15.10 (Gluing near a closed set [GN26, Lemma A.5.10])**.** *Let Z be paracompact, $A \subseteq Z$ closed, (V_i) open sets covering A, and g_i maps such that g_i and g_j agree near each point of $A \cap V_i \cap V_j$. Then there are an open $U \supseteq A$ and a map g which near each point of U agrees with some g_i on an open subset of V_i.*

Proof. As in [GN26]: closed neighborhoods $Q_z \subseteq V_{i(z)}$ of the points of A, a locally finite open refinement (W_j) of the cover by their interiors and $Z \setminus A$, and around each $a \in A$ a neighborhood meeting finitely many W_j, avoiding the closures of those not containing a in their closure, and contained in the sets where the corresponding g_i agree. □

Theorem 15.11 (Extension to complexifications [GN26, Proposition 2.2.19])**.** *Let M^* be a complexification of the real analytic manifold M, with M closed in M^* and M^* paracompact. Every real analytic map from M to a complex analytic manifold L extends to a complex analytic map on an open neighbourhood of M in M^*.*

Proof. In a chart φ of M^* restricting to a chart ψ of M and a chart κ of L, the local representative $\kappa \circ f \circ \psi^{-1}$ has a complex analytic extension, which after shrinking has real points only where f is defined in these charts; this gives local extensions g_x. Two of them agree near each point of M: on a convex neighbourhood in $E_{\mathbb{C}}$ their representatives agree on the real points, hence everywhere by the identity theorem. Lemma 15.10 glues them. □

15.4 Real analytic forms

Definition 15.12 (Real analytic forms)**.** Let M be a real analytic manifold. A form $\omega \in \Omega^n(M, F)$ is *real analytic* if its local representatives (Definition 8.7) are real analytic in the sense of Milnor [GN26, Remark 3.6.18]. The real analytic forms are a subspace $\Omega^n_\omega(M, F)$ of $\Omega^n(M, F)$. A smooth vector field is real analytic if its local representatives are.

Theorem 15.13 (Operations on real analytic forms)**.** *Exterior derivatives, compositions with continuous linear maps, wedge products and pullbacks along real analytic maps of real analytic forms are real analytic. Brackets of real analytic vector fields are real analytic, and so are $i_X\omega$ and $L_X\omega$ for real analytic X and ω. Hence all the identities of Section 8.6 hold in $\Omega^\bullet_\omega(M, F)$, as asserted in [GN26, Remark 3.6.18].*

Proof. In a chart, each operation is built from the local representatives by sums, compositions with continuous (multi)linear maps, compositions with real analytic maps and directional derivatives. Real analytic maps are closed under all of these, the last one because if g is a complex analytic extension of f, then $\mathrm{d}g$, read on $(E \times E)_{\mathbb{C}} \cong E_{\mathbb{C}} \times E_{\mathbb{C}}$, is a complex analytic extension of $\mathrm{d}f$ (`LocallyConvex.Analysis.RealAnalyticOn.dMB` in `lc-analysis`). For pullbacks, real analyticity is local [GN26, Lemma 2.2.3], and near each point the representative of $\varphi^*\omega$ is a local pullback along a real analytic map between chart domains. □

Definition 15.14 (Real analytic de Rham cohomology)**.** Let $Z^n_\omega(M, F)$ be the closed real analytic n-forms and $B^n_\omega(M, F) = \mathrm{d}(\Omega^{n-1}_\omega(M, F))$, with $B^0_\omega = 0$. The real analytic de Rham cohomology is $H^n_\omega(M, F) = Z^n_\omega(M, F)/B^n_\omega(M, F)$, and there is a natural linear map $H^n_\omega(M, F) \to H^n_{dR}(M, F)$.

Remark 15.15. For a real analytic submanifold V of $\mathbb{R}^n$ this map is an isomorphism [Car57, §7, consequence (4), p. 89], a consequence of Cartan's Theorem B for coherent analytic sheaves on analytic subsets of $\mathbb{R}^n$ [Car57, Theorem 3]; the proof is omitted there. This covers every second countable real analytic manifold, since such a manifold admits a real analytic, regular and proper embedding into some $\mathbb{R}^k$ [Gra58, Theorem 3, p. 470]. The proof uses complexifications and sheaf cohomology on Stein manifolds, with no counterpart for general locally convex models. We do not know whether it is injective or surjective in general.

Remark 15.16 (What is not covered). The following parts of [GN26, Section 3.6] are not formalized.

- The finite-dimensional material (Definition 3.6.19 to Remark 3.6.24: forms on subsets of $\mathbb{R}^n$, orientations, integration of forms and Stokes' Theorem).

Chapter 16

Nijenhuis operators, complex and Kähler manifolds

Complex analytic manifolds are introduced in Chapter 15. This chapter studies fields of endomorphisms of the tangent bundle, their Nijenhuis torsion, almost complex structures and Kähler structures. As for metrics and forms, fields of endomorphisms are stored in the coordinates of the preferred chart at each point, and their smoothness is that of their local representatives. We follow the conventions of [GLT].

Definition 16.1 (Fields of endomorphisms)**.** A *smooth field of endomorphisms* N of TM is a field of real linear maps N_x of the tangent spaces whose local representatives $(y, v) \mapsto N_i(y)v$ are smooth. It acts on vector fields, $(NX)(x) = N_x X(x)$, and the local representative of NX is $N_i X_i$.

Definition 16.2 (The Nijenhuis torsion)**.** The *Nijenhuis torsion* of N is $\Omega_N(X, Y) = N[NX, Y] + N[X, NY] - [NX, NY] - N^2[X, Y]$ [GLT, Definition 3.1]. In a chart it is $\Omega_N(u, v) = N((\partial_u N)v - (\partial_v N)u) + (\partial_{Nv} N)u - (\partial_{Nu} N)v$.

Theorem 16.3 (The Nijenhuis torsion is a tensor)**.** *For C^1 local vector fields X, Y, $N[NX, Y] + N[X, NY] - [NX, NY] - N^2[X, Y] = \Omega_N(X, Y)$, with Ω_N given by the local formula of Definition 16.2. In particular, for C^1 vector fields on M, the torsion at x only depends on $X(x)$ and $Y(x)$ [GLT, Theorem 3.2].*

Proof. Expand the brackets with the product rule $\mathrm{d}(NX)(w) = (\partial_w N)X + N\,\mathrm{d}X(w)$: all the derivatives of X and Y cancel. For vector fields on M, compute in the preferred chart at x, where the local representative of NX is $N_i X_i$. □

Definition 16.4 (Almost complex structures)**.** An *almost complex structure* is a smooth field of endomorphisms J with $J^2 = -1$. It is *integrable* if its Nijenhuis torsion vanishes in every chart; then $J[JX, Y] + J[X, JY] - [JX, JY] + [X, Y] = 0$ for C^1 vector fields.

Theorem 16.5 (The almost complex structure of a complex manifold)**.** *On a complex analytic manifold, multiplication by i in the charts is a well defined almost complex structure, and it is integrable.*

Proof. The transition maps are complex analytic, so their derivatives are complex linear ([LCA], `complexAnalyticOn_iff_holomorphicOn`); hence the local representatives of J are multiplication by i in every chart. Being constant, they have vanishing Nijenhuis torsion. □

The converse is the Newlander–Nirenberg theorem [NN57]. In infinite dimensions it holds for real analytic almost complex structures on real analytic Banach manifolds [Bel05, Theorem 7, p. 64] (Theorem 16.7 below), but it fails for smooth structures on some Banach manifolds [Pat99, Theorem 3.1, p. 401]; for Fréchet manifolds, Lempert points out loop spaces with integrable almost complex structures that are not locally biholomorphic to locally convex spaces [Lem98, §2, pp. 489–490].

Definition 16.6 (Real analytic structures and holomorphic charts)**.** An almost complex structure J is real analytic if its local representative in every chart is a real analytic map into $L(E)$. A J-holomorphic chart is a chart ψ into a complex normed space, smooth with smooth inverse in the real sense, with $D\psi(Jv) = i\,D\psi(v)$.

Theorem 16.7 (Newlander–Nirenberg, real analytic case)**.** *Let M be modeled on a real Banach space and J an integrable real analytic almost complex structure on M. Then every point lies in the domain of a J-holomorphic chart into $\ker(J(x_0)_{\mathbb{C}} - i)$, and the transition maps between J-holomorphic charts are complex analytic [Pen70, Théorème 3.5].*

Proof. In the preferred chart at x_0, J is a real analytic $\hat{J}$ with $\hat{J}^2 = -1$ and vanishing Nijenhuis tensor; the local theorem of [LCA] gives a chart φ with $D\varphi \circ \hat{J} = i\,D\varphi$, and $\psi = \varphi \circ$ chart is J-holomorphic. For two such charts, the real derivative of $\psi_2 \circ \psi_1^{-1}$ is $B \circ A$, where A intertwines i and J and B intertwines J and i; so it is complex linear, and the transition map is complex differentiable, hence complex analytic. □

Definition 16.8 (Hermitian structures and the fundamental form)**.** A *Hermitian structure* is an almost complex structure J together with a smooth field of symmetric bilinear forms g, not necessarily positive, with $g(Ju, Jv) = g(u, v)$. Its *fundamental form* is the 2-form $\omega(u, v) = g(Ju, v)$.

Theorem 16.9 (Kähler structures are symplectic)**.** *A* Kähler structure *is a Hermitian structure with J integrable, g weakly nondegenerate and ω closed. Its fundamental form is a weak symplectic form.*

Proof. If $\omega(v, \cdot) = g(Jv, \cdot) = 0$, then $Jv = 0$ by weak nondegeneracy of g, so $v = -J^2 v = 0$. Integrability is not used. □

Example 16.10 (The flat Kähler structure). A complex inner product space E, not necessarily complete, carries the Kähler structure $J = i$, $g = \mathrm{Re}\langle \cdot, \cdot \rangle$, with fundamental form $\omega(u, v) = \mathrm{Im}\langle u, v \rangle$.

Chapter 17

Symplectic manifolds

Symplectic forms on manifolds modeled on locally convex spaces. In infinite dimensions nondegeneracy splits into a weak and a strong notion, and many symplectic forms of interest are only weakly nondegenerate. Hamiltonian vector fields are then unique but need not exist, and the Darboux theorem fails [Mar72, Proposition, p. 591]. For strong symplectic forms on Banach manifolds Darboux's theorem holds [Wei69, §1, Theorem]; Moser's proof needs flows of time-dependent vector fields, which exist on Banach manifolds but need not exist on locally convex ones, and it is formalized in Section 17.7 for Banach models. Real-valued forms require models in the lowest universe, since the space of values of a form lives in the universe of the model.

17.1 Symplectic forms

Definition 17.1 (Symplectic forms)**.** A *symplectic form* on M is a closed 2-form $\omega \in \Omega^2(M, \mathbb{R})$ which is *weakly nondegenerate*: if $\omega_x(v, w) = 0$ for all $w \in T_xM$, then $v = 0$. It is *strong* if moreover every continuous linear functional on T_xM is of the form $\omega_x(v, \cdot)$, for every $x \in M$.

In other words, ω is weak if the flat map $\omega_x^\flat \colon T_xM \to T_xM'$, $v \mapsto \omega_x(v, \cdot)$, is injective, and strong if it is bijective. We use no topology on the dual space.

Theorem 17.2 (Strong symplectic normed spaces are reflexive)**.** *Let V be a normed space and ω a continuous skew-symmetric bilinear form on V such that $\omega^\flat \colon V \to V'$ is surjective. Then V is reflexive: the canonical map $V \to V''$ is surjective. Consequently, a nonempty manifold modeled on a normed space E which carries a strong symplectic form has a reflexive model.*

Proof. Let $\Phi \in V''$. The functional $-\Phi \circ \omega^\flat$ is continuous, so it equals $\omega(v, \cdot)$ for some $v \in V$. For $\varphi = \omega(w, \cdot) \in V'$, $\varphi(v) = \omega(w, v) = -\omega(v, w) = \Phi(\omega^\flat w) = \Phi(\varphi)$, and every φ is of this form. For the second statement, ω_x is continuous on $E \times E$ because its local representative is, so it is a bounded bilinear form on the normed space E. No completeness is used. □

In particular non-reflexive Banach spaces such as ℓ^1 or c_0 do not model strong symplectic manifolds. Marsden observes the converse phenomenon for cotangent bundles: the canonical form of T^*M is strong if the model of M is reflexive, and only weak otherwise [Mar72, p. 590].

17.2 Hamiltonian and symplectic vector fields

Definition 17.3 (Hamiltonian and symplectic vector fields)**.** A smooth vector field X is a *Hamiltonian vector field* of $f \in C^\infty(M, \mathbb{R})$ if $i_X\omega = \mathrm{d}f$, and a *symplectic vector field* if $L_X\omega = 0$.

Theorem 17.4 (Hamiltonian and symplectic fields)**.** *Let ω be a symplectic form on M.*

1. *A vector field is determined by $i_X\omega$; in particular Hamiltonian vector fields are unique.*
2. *X is symplectic if and only if $i_X\omega$ is closed. Hamiltonian fields are symplectic.*
3. *If M is smoothly contractible, every symplectic vector field is Hamiltonian.*
4. *If X and Y are symplectic, then $i_{[X,Y]}\omega = \mathrm{d}(\omega(Y,X))$; so $[X,Y]$ is Hamiltonian.*
5. *(Conservation of energy) If X is the Hamiltonian field of f, then $L_X f = 0$.*

Proof. (1) is weak nondegeneracy at each point. (2) Since ω is closed, Cartan's formula gives $L_X\omega = \mathrm{d}(i_X\omega)$; and $\mathrm{d}\mathrm{d}f = 0$. (3) is the Poincaré Lemma applied to the closed 1-form $i_X\omega$. (4) From $[L_X, i_Y] = i_{[X,Y]}$ and $L_X\omega = 0$ we get $i_{[X,Y]}\omega = L_X i_Y\omega = \mathrm{d}i_X i_Y\omega + i_X\mathrm{d}i_Y\omega$, and $\mathrm{d}i_Y\omega = 0$. (5) $L_X f = i_X\mathrm{d}f = i_X i_X\omega = 0$. □

Definition 17.5 (Poisson bracket)**.** For f with Hamiltonian vector field X_f and any $g \in C^\infty(M,\mathbb{R})$, the *Poisson bracket* is $\{f,g\} = -L_{X_f}g$.

Theorem 17.6 (Properties of the Poisson bracket)**.** *Let f, g, h have Hamiltonian vector fields X_f, X_g, X_h. Then $\{f,g\} = \omega(X_f, X_g) = L_{X_g}f = -\{g,f\}$; the bracket $[X_f, X_g]$ is the Hamiltonian vector field of $-\{f,g\}$; and the Jacobi identity $\{f,\{g,h\}\} - \{g,\{f,h\}\} = \{\{f,g\},h\}$ holds.*

Proof. The first identities follow from $i_{X_g}\omega = \mathrm{d}g$ and the skew-symmetry of ω, and the statement on $[X_f, X_g]$ is part (4) of Theorem 17.4. For the Jacobi identity, $[L_{X_f}, L_{X_g}] = L_{[X_f,X_g]}$ on functions, and $L_{[X_f,X_g]}h = i_{[X_f,X_g]}i_{X_h}\omega = -i_{X_h}i_{[X_f,X_g]}\omega = -L_{X_h}L_{X_f}g$. □

17.3 Symplectic maps

Definition 17.7 (Symplectic maps)**.** A smooth map $\varphi\colon (N,\omega_0) \to (M,\omega)$ is *symplectic* if $\varphi^*\omega = \omega_0$.

Proposition 17.8 (Symplectic maps)**.** *Identities and compositions of symplectic maps are symplectic. If φ is symplectic, X and Y are φ-related and Y is the Hamiltonian field of f, then X is the Hamiltonian field of $f \circ \varphi$.*

Proof. $\varphi^*(i_Y\omega) = i_X\varphi^*\omega$ and $\varphi^*\mathrm{d}f = \mathrm{d}(f\circ\varphi)$. □

17.4 The canonical symplectic form

Definition 17.9 (Canonical symplectic form)**.** Let H be a real inner product space, not necessarily complete. The *Liouville form* on $H \times H$ is $\theta_{(x,a)}(y,b) = \langle x, b\rangle$, and the *canonical symplectic form* is $\omega = \mathrm{d}\theta$, $\omega((y,b),(y',b')) = \langle y, b'\rangle - \langle y', b\rangle$. This is the skew form induced by the symmetric form $\langle\cdot,\cdot\rangle$ in [Mar72, p. 590].

Proposition 17.10 (The canonical symplectic form)**.** *The canonical symplectic form is exact and weakly nondegenerate. If H is a Hilbert space, it is strong.*

Proof. If $\omega((y,b),\cdot) = 0$, evaluating at $(0,y)$ and $(b,0)$ gives $y = b = 0$. If H is complete, a functional φ on $H \times H$ is $\varphi(y',b') = \langle a_1, y'\rangle + \langle a_2, b'\rangle$ by the Riesz representation theorem, and $\varphi = \omega((a_2, -a_1),\cdot)$. □

If H is not complete, then $H \times H$ is not reflexive (its bidual is the bidual of the completion), so by Theorem 17.2 it carries no strong symplectic form at all; this converse is not formalized.

17.5 The real analytic Darboux theorem

Real analyticity is in the sense of Milnor (Definition 2.2.2 of [GN26]): restrictions of complex analytic maps on open subsets of the complexification.

Lemma 17.11 (A real analytic primitive)**.** *Let E and $D \cong E'$ be Banach spaces and $\Theta\colon B(p,r) \to \mathcal{L}(E,D)$ real analytic, skew-symmetric and closed, with $\Theta(p) = 0$. On a smaller ball, $\alpha(y) = \int_0^1 s\,\Theta(p + s(y-p))(y-p)\,\mathrm{d}s$ is real analytic, $\mathrm{d}\alpha = \Theta$, $\alpha(p) = 0$ and $\mathrm{d}\alpha(p) = 0$.*

Proof. Differentiating under the integral sign, $\mathrm{d}\alpha(y)(a,b)$ is the integral of $2s\,\Theta(z_s)(a,b) + s^2\,\mathrm{d}\Theta(z_s)(y-p)(a,b) = \frac{\mathrm{d}}{\mathrm{d}s}(s^2\Theta(z_s)(a,b))$, $z_s = p + s(y-p)$, by the closedness of Θ and the skew-symmetry of $\mathrm{d}\Theta$. For real analyticity, $\alpha(y)$ is the value at time 1 of the solution of $\dot{y} = 0$, $\dot{a} = s\,\Theta(p + s(y-p))(y-p)$ starting at $(y,0)$; the time-dependent field is real analytic and stationary at $(p,0)$, and the flow up to time 1 near $(p,0)$ is real analytic (Theorem 5.4 and its real analytic version). This replaces real analytic parametric integrals, which are not available. □

Theorem 17.12 (The real analytic local Darboux theorem)**.** *In Theorem 17.18, if $\Omega\colon U \to \mathcal{L}(E,E')$ is real analytic, then ψ and ψ^{-1} can be chosen real analytic.*

Proof. Moser's method with the primitive of Lemma 17.11. The field $X_t = -\Omega_t^{-1}\alpha$ is real analytic since inversion in the Banach algebra $\mathcal{L}(E)$ is [GN26, Proposition 2.2.13]; the flow of $(1, X_t)$ is real analytic, so $\psi = F_1$ is; and $\psi^{-1}(y) = \mathrm{Fl}(-1,(1,y))$ for the flow of the suspended field, so no real analytic inverse function theorem is needed. We do not claim this argument is new; we know no reference for the real analytic case on Banach manifolds. □

Theorem 17.13 (Real analytic Darboux charts)**.** *Let M be a real analytic manifold modeled on a Banach space and ω a closed real analytic real 2-form [GN26, Remark 3.6.18], strongly nondegenerate at x_0. Then the Darboux chart of Theorem 17.19 can be chosen in the real analytic maximal atlas.*

Proof. As for Theorem 17.19, with Theorem 17.12; $\varphi = \psi^{-1} \circ i$ is compatible with the real analytic atlas since ψ is a real analytic diffeomorphism. □

Remark 17.14 (On the hypothesis). Real analytic forms are defined by the real analyticity of $(y,u,v) \mapsto \Omega(y)(u,v)$ [GN26, Remark 3.6.18], while the local theorem needs $y \mapsto \Omega(y) \in \mathcal{L}(E,E')$ real analytic for the operator norm. The second follows from the first: this is the real analytic counterpart of the theorem on operator-valued maps used in Theorem 9.5, proved in lc-analysis by extending to the complexification and applying the holomorphic version there.

17.6 The cotangent bundle

In this section E is a normed space (not necessarily complete) and M a manifold modeled on E, so that the cotangent bundle $T^*M = (TM)'$ exists, with fiber $E' = \mathcal{L}(E,\mathbb{R})$ (Theorem 9.5); its total space is a manifold modeled on $E \times E'$.

Definition 17.15 (The canonical 1-form [Lan99, §V.7, Proposition 7.1])**.** The *canonical (Liouville) 1-form* θ on T^*M is $\theta_\lambda(z) = \lambda(T\pi\, z)$, for the projection $\pi\colon T^*M \to M$. It is a smooth 1-form: in the chart of T^*M built from a chart c of M and the trivialization over a chart j, its local representative is $((y,\mu),(u,\psi)) \mapsto \mu(g_{cj}(y)u)$, where $g_{cj}(y) = \mathrm{d}(j \circ c^{-1})(y)$; for $j = c$ it is $\theta_{(y,\mu)}(u,\psi) = \mu(u)$.

Proposition 17.16 (The canonical symplectic form [Lan99, §V.7, Proposition 7.2])**.** *The canonical symplectic form $\omega = -\mathrm{d}\theta$ on T^*M is $\omega_\lambda((u_1,\psi_1),(u_2,\psi_2)) = \psi_2(u_1) - \psi_1(u_2)$ in the preferred chart. It is exact and weakly nondegenerate, for every normed model E.*

Proof. In the preferred chart θ is $((y,\mu),(u,\psi)) \mapsto \mu(u)$, linear in (y,μ), and the local formula for d gives $\mathrm{d}\theta((u_1,\psi_1),(u_2,\psi_2)) = \psi_1(u_2) - \psi_2(u_1)$. If $\omega((u_1,\psi_1),\cdot) = 0$, testing on $(0,\psi)$ gives $\psi(u_1) = 0$ for all $\psi \in E'$, so $u_1 = 0$ by the Hahn–Banach theorem, and testing on $(u,0)$ gives $\psi_1 = 0$. We follow Lang's sign, which agrees with Definition 17.9 under $\psi \leftrightarrow \langle\cdot, b\rangle$. □

Theorem 17.17 (Strong nondegeneracy and reflexivity)**.** *If M is nonempty, the canonical symplectic form on T^*M is strongly nondegenerate if and only if E is reflexive, i.e. the canonical map $E \to E''$ is surjective. In particular the cotangent bundle of a manifold modeled on a non-reflexive Banach space (such as ℓ^1 or c_0) is only a weak symplectic manifold; cotangent bundles of non-reflexive spaces are mentioned in [Mar72, p. 590].*

Proof. A functional φ on $E \times E'$ is $\omega((u,\psi),\cdot)$ for $\psi = -\varphi(\cdot,0)$ and $u \in E$ representing $\varphi(0,\cdot) \in E''$. Conversely, for $\Phi \in E''$, write $\Phi \circ \mathrm{pr}_2 = \omega((u,\psi),\cdot)$ and test on $(0,\psi')$: $\Phi(\psi') = \psi'(u)$. No completeness is used. (Theorem 17.2 would give the reflexivity of $E \times E'$.) □

17.7 The Darboux theorem

In this section E is a Banach space. A closed real 2-form ω is *strongly nondegenerate at x_0* if $\omega^\flat_{x_0} : T_{x_0}M \to T_{x_0}M'$ is bijective.

Theorem 17.18 (The local Darboux theorem [Wei69, §1, Theorem], [Lan99, Theorem V.8.1])**.** *Let $U \subseteq E$ be open and $\Omega\colon U \to \mathcal{L}(E,E')$ smooth for the operator norm, with $\Omega(y)$ skew-symmetric, $\mathrm{d}\Omega = 0$, and $\Omega(p)$ bijective. Then there is a diffeomorphism ψ between open neighborhoods of p in U with $\psi(p) = p$, $\mathrm{d}\psi(p) = \mathrm{id}$ and $\psi^*\Omega = \Omega(p)$: $\Omega(\psi x)(\mathrm{d}\psi(x)u, \mathrm{d}\psi(x)v) = \Omega(p)(u,v)$.*

Proof. Moser's method [Mos65, §4, Theorem 2]. Put $\Omega_0 = \Omega(p)$, $\Theta = \Omega - \Omega_0$ and $\Omega_t = \Omega_0 + t\Theta$.

1. On a ball B around p, $\Theta = \mathrm{d}\alpha$ with $\alpha(p) = 0$ and $\mathrm{d}\alpha(p) = 0$: the Poincaré lemma for star-shaped sets (Corollary 8.37) gives a primitive η, from which we subtract the closed affine form $\eta(p) + \mathrm{d}\eta(p)(y-p)$.

2. The set W of (t,y) with $\Omega_t(y)$ invertible is open and contains $\mathbb{R} \times \{p\}$. On W put $X_t(y) = -\Omega_t(y)^{-1}\alpha(y)$, so that $\Omega_t(X_t,\cdot) = -\alpha$, $X_t(p) = 0$ and $\partial_x X_t(p) = 0$.

3. The autonomous field $Z(t,y) = (1, X_t(y))$ on the open set $W \subseteq \mathbb{R} \times E$, regarded as a manifold, has the integral curve $t \mapsto (t,p)$ through $(0,p)$, defined on $\mathbb{R}$. Since the domain of the maximal flow is open (Theorem 5.4) and $[-1,2]$ is compact, the flow F_t of X_t with $F_0 = \mathrm{id}$ is defined and smooth on $(-\delta, 1+\delta) \times V$ for a neighborhood V of p; this is Weinstein's "compactness of $[0,1]$ and openness of the domain".

4. For $x \in V$ and $u,v \in E$, $t \mapsto \Omega_t(F_t x)(\partial_x F_t(x)u, \partial_x F_t(x)v)$ has derivative 0, by the variational equation $\partial_t \partial_x F_t = \partial_x X_t \circ \partial_x F_t$ and an algebraic identity, the pointwise form of $\frac{\mathrm{d}}{\mathrm{d}t}(F_t^*\Omega_t) = F_t^*(\mathrm{d}\alpha + \mathrm{d}(i_{X_t}\Omega_t)) = 0$.

5. Hence $F_1^*\Omega = \Omega_0$, and $\partial_x F_t(p) = \mathrm{id}$, so by the inverse function theorem $\psi = F_1$ is a local diffeomorphism at p.

No bump functions are used (they need not exist on a Banach space): the field lives on W and is never cut off. We normalize the primitive so that $\mathrm{d}\alpha(p) = 0$, which gives $\mathrm{d}\psi(p) = \mathrm{id}$. We do not claim this argument is new. □

Theorem 17.19 (Darboux charts [Wei69, §1, Theorem], [Lan99, Theorem V.8.1])**.** *Let M be modeled on a Banach space and ω a closed real 2-form on M, strongly nondegenerate at x_0. There is a chart φ of the maximal atlas around x_0, with the same value and differential at x_0 as the preferred chart, in which ω has constant coefficients, equal to its value at x_0. In particular every strong symplectic manifold modeled on a Banach space is locally isomorphic to its model with a constant form.*

Proof. In the preferred chart i at x_0, ω is a smooth map $y \mapsto \Omega(y) \in \mathcal{L}(E, E')$, continuous for the operator norm since the model is normed (lc-analysis). Apply Theorem 17.18 and put $\varphi = \psi^{-1} \circ i$. □

Remark 17.20 (Infinite-dimensional subtleties).

- Strong nondegeneracy is used once, to define X_t, and only at x_0: with completeness the open mapping theorem makes $\Omega_0 \colon E \to E'$ a topological isomorphism, and invertible operators form an open set for the operator norm. For weakly nondegenerate forms the theorem fails, even on Hilbert space [Mar72, Proposition, p. 591]. Marsden's counterexample is formalized in lc-analysis (`LocallyConvex.Analysis.Counterexampl`
- Reflexivity is a consequence, not a hypothesis (Theorem 17.2).
- Completeness of E is used for the flows, the inverse function theorem, the smoothness of $L \mapsto L^{-1}$ and the open mapping theorem. On non-normable models there is no vector topology on $\mathcal{L}(E, E')$ for which evaluation is continuous, and neither the flows nor the inverse function theorem are available.
- No Hausdorff hypothesis on M is needed: the flows live in $W \subseteq \mathbb{R} \times E$ and the argument takes place in one chart.

Weinstein also proves a relative version for pairs of manifolds [Wei71, Theorem 4.1, Corollary 4.2], which is not formalized.

Chapter 18

Examples

In this chapter we test the general notions of the previous chapters on concrete examples: closed subspaces, open subsets, graphs, the unit sphere of an inner product space and the unitary group of a Banach $*$-algebra. They show that embedded and split submanifolds, fields of projectors and their geodesics behave as expected; in all of them the slice charts are explicit, so that no inverse function theorem is needed. The last section prepares the ground for groups of diffeomorphisms: we construct the Fréchet space C^∞_{per} of smooth periodic functions and identify the universal cover of $\mathrm{Diff}^+(S^1)$ with an open convex subset of it.

Throughout, we keep track of completeness. A model space is a real locally convex Hausdorff space, not necessarily complete; when an example is a Banach, Hilbert or Fréchet manifold we say so, and we say where completeness is used.

18.1 Closed subspaces, open subsets and graphs

The simplest submanifolds are linear. Let F be a locally convex space and $S \subseteq F$ a linear subspace.

Proposition 18.1 (Closed subspaces)**.** *If S is closed, then S is an embedded submanifold of F modeled on S, with the identity of F as slice chart at every point, and $T_pS = S$ for every $p \in S$. If S is complemented, S is a split submanifold.*

Proof. The identity belongs to the maximal atlas of F, and $q \in S$ if and only if $\mathrm{id}(q) \in S$. The differential of the identity is the identity, so the tangent space read in this chart is S. □

Proposition 18.2 (Linear inclusions)**.** *If S is closed, the inclusion $\iota\colon S \to F$ is an immersion. It is a split immersion if and only if S is complemented in F.*

Proof. In the identity charts the differential of ι at every point is ι itself, which is a topological isomorphism onto the closed subspace S. A continuous left inverse of ι is the same thing as a continuous projection of F onto S. □

A continuous projection P of F onto S ($P^2 = P$, $\operatorname{ran} P = S$) gives the simplest field of projectors: the constant one.

Proposition 18.3 (The flat connection of a subspace)**.** *Let S be closed and P a continuous projection onto S. Then $E_p = P$ for all $p \in S$ is a field of projectors, the symbols of its connection vanish in the identity chart, and a C^2 curve α in S is a geodesic if and only if $\ddot\alpha(t) = 0$ for all t.*

Proof. Smoothness of $(p, w) \mapsto Pw$ is that of a continuous linear map. The symbols involve the second differential of the inclusion, which is zero. For the geodesics, the general criterion says that α is a geodesic iff $P\ddot{\alpha} = 0$; since $\ddot{\alpha}$ takes values in S and P is the identity on S, this is $\ddot{\alpha} = 0$. □

So the connection is flat and its geodesics are the curves of zero acceleration, that is, the affine lines $t \mapsto x + tv$ (this last reformulation is the usual one and is not stated separately in Lean).

Proposition 18.4 (Open subsets)**.** *An open subset U of a manifold M modeled on E is an embedded submanifold of M, modeled on the whole space E.*

Proof. The restriction of the chart at p to U is again a chart (the Michal–Bastiani groupoid is closed under restriction to open sets), and every point of its domain lies in U. □

Let now E, G be locally convex spaces, $U \subseteq E$ open and $g\colon U \to G$ smooth. The graph of g is

$$\Gamma_g = \{(x, y) \in E \times G : x \in U,\ y = g(x)\},$$

and its model is the horizontal subspace $E \times 0$, the kernel of the second projection.

Proposition 18.5 (Graphs)**.** *The map $\Phi(x, y) = (x, y - g(x))$, defined on $U \times G$, is a chart of $E \times G$ with inverse $\Phi^{-1}(x, z) = (x, z + g(x))$, and $(x, y) \in \Gamma_g$ if and only if $\Phi(x, y) \in E \times 0$. Hence Γ_g is a split submanifold of $E \times G$ modeled on $E \times 0$.*

Proof. Both Φ and Φ^{-1} are built from g, projections and sums, hence they are smooth on $U \times G$. The subspace $E \times 0$ is closed and complemented by $0 \times G$, with projection $(x, y) \mapsto (x, 0)$. □

Remark 18.6. In finite dimensions one usually proves that graphs (and level sets of submersions) are submanifolds through the inverse function theorem. That theorem is not available for Michal–Bastiani maps between locally convex spaces, and it is not needed here: the chart Φ and its inverse are written down explicitly. The same strategy is used for the sphere and the unitary group below.

18.2 The unit sphere of an inner product space

Let H be a real inner product space and $e \in H$ a unit vector. We write $e^{\perp} = \{w : \langle w, e\rangle = 0\}$, a closed subspace of H, and $S = \{q \in H : \|q\| = 1\}$.

Remark 18.7 (Completeness). We do not assume that H is complete. If H is complete, H is a Hilbert manifold and S is a Hilbert submanifold. If H is not complete, H is still a manifold in our sense (model spaces need not be complete), but neither H nor S is a Hilbert or Banach manifold. The construction below does not use completeness, so it covers both cases; the statements about Hilbert manifolds are the case of a complete H.

On the open hemisphere around e we use the explicit chart

$$\varphi_e(q) = q - \sqrt{1 - \|q - \langle q, e\rangle e\|^2}\, e, \qquad \varphi_e^{-1}(r) = r + \sqrt{1 - \|r - \langle r, e\rangle e\|^2}\, e,$$

defined on $\{q : \|q - \langle q, e\rangle e\| < 1,\ \langle q, e\rangle > 0\}$. By Pythagoras, $\|q\|^2 = \|q - \langle q, e\rangle e\|^2 + \langle q, e\rangle^2$, so on this set $\|q\| = 1$ exactly when the height $\sqrt{1 - \|q - \langle q, e\rangle e\|^2}$ equals $\langle q, e\rangle$, that is, when $\varphi_e(q) \in e^{\perp}$. At another point $p \in S$ we first apply the Householder reflection R_p exchanging p and e (the reflection across $(p - e)^{\perp}$), a linear isometry, and then φ_e.

Lemma 18.8 (The charts of the sphere)**.** *The map φ_e is a smooth chart of H with smooth inverse; for q in its domain, $\|q\| = 1$ if and only if $\varphi_e(q) \in e^\perp$. For every unit vector p, the chart $\varphi_e \circ R_p$ is smooth and its domain contains p.*

Proof. The square root is smooth where its argument is positive, and the remaining operations are continuous linear or bilinear; Fréchet smoothness implies Michal–Bastiani smoothness. The identity $\varphi_e^{-1}\varphi_e = \mathrm{id}$ holds because the component of q orthogonal to e, and hence the height, does not change when we add a multiple of e. The equivalence is Pythagoras, and $R_p p = e$ lies in the domain of φ_e. □

Theorem 18.9 (The sphere is an embedded submanifold)**.** *The unit sphere S is an embedded submanifold of H modeled on $e^\perp$, and $T_pS = p^\perp$ for every $p \in S$.*

Proof. Since R_p is an isometry, $\|q\| = 1$ iff $\|R_p q\| = 1$ iff $\varphi_e(R_p q) \in e^\perp$. At e the height has zero derivative, so $\mathrm{d}\varphi_e(e) = \mathrm{id}$, and by the chain rule $\mathrm{d}(\varphi_e \circ R_p)(p) = R_p$. Then $w \in T_pS$ iff $\langle e, R_p w\rangle = 0$ iff $\langle R_p p, R_p w\rangle = 0$ iff $\langle p, w\rangle = 0$. □

The orthogonal projections onto the tangent spaces give a field of projectors, and the geodesics of its connection are the curves whose acceleration is normal to the sphere.

Theorem 18.10 (Geodesics of the sphere)**.** *The maps $E_p w = w - \langle w, p\rangle p$, $p \in S$, form a field of projectors onto the tangent spaces of S. A curve α in S whose ambient curve is smooth is a geodesic of the associated connection if and only if*

$$\ddot\alpha(t) = \langle \ddot\alpha(t), \alpha(t)\rangle\, \alpha(t) \quad \text{for all } t.$$

In particular, if $p, v \in H$ are unit vectors with $\langle p, v\rangle = 0$, the great circle $t \mapsto \cos t\, p + \sin t\, v$ is a geodesic.

Proof. It is immediate that E_p is linear, idempotent, with range $p^\perp$. In a slice chart, $(x, w) \mapsto E_{\psi(x)} w$ is smooth because the parametrization ψ is. By the general criterion, α is a geodesic iff $E_{\alpha(t)}\ddot\alpha(t) = 0$, which is the stated equation. For the great circle, $\ddot\alpha = -\alpha$ and $\|\alpha\| = 1$. □

Theorem 18.11 (A local addition on the sphere)**.** *The radial projection $\Sigma(p, w) = (p + w)/\|p + w\|$, $w \perp p$, is a normalized local addition on S, defined on all of TS. The map $\theta = (\pi, \Sigma)$ sends TS onto $\{(p, q) \in S \times S : \langle p, q\rangle > 0\}$, with inverse $(p, q) \mapsto (p, q/\langle p, q\rangle - p)$. No completeness is assumed.*

Proof. Apply Proposition 14.6 with $\sigma(p, w) = (p + w)/\|p + w\|$ on $p + w \neq 0$ and $\tau(p, q) = q/\langle p, q\rangle - p$ on $\langle p, q\rangle > 0$, both smooth in the Fréchet sense, hence in the sense of Michal–Bastiani. For $\|p\| = 1$ and $w \perp p$, $\langle p, \sigma(p, w)\rangle = \|p + w\|^{-1} > 0$ and $\tau(p, \sigma(p, w)) = w$; conversely $\tau(p, q) \perp p$ and $\sigma(p, \tau(p, q)) = q$. The derivative of $t \mapsto \sigma(p, tw)$ at 0 is $w - \langle p, w\rangle p = w$. Glöckner and Neeb do not treat the sphere; we do not claim this construction is new. □

Theorem 18.12 (The sphere is real analytic, with a real analytic local addition)**.** *Let H be a real inner product space, not necessarily complete, and $e \in H$ a unit vector. The unit sphere S is a split real analytic submanifold of H modeled on $e^\perp$, hence a real analytic manifold, and the radial projection $\Sigma(p, w) = (p + w)/\|p + w\|$ of Theorem 18.11 is a real analytic local addition on it [GN26, p. 325].*

Proof. The inner product is a continuous bilinear form, hence real analytic; its complexification is the complex bilinear (not sesquilinear) extension to $H_{\mathbb{C}} \times H_{\mathbb{C}}$. The functions $t \mapsto \sqrt{t}$ on $t > 0$ and $t \mapsto t^{-1}$ on $t \neq 0$ extend holomorphically to the slit plane and to $\mathbb{C}\setminus\{0\}$. The norm is used only as

$\|x\| = \sqrt{\langle x, x\rangle}$ on $x \neq 0$. So σ, τ and the charts of the sphere (built from the height $\sqrt{1 - \|q'\|^2}$, with q' the component of q in $e^\perp$, and from Householder reflections) are real analytic, the model $e^\perp$ is complemented by the orthogonal projection, and Proposition 14.15 applies. Normalization of the real analytic local addition is not proved; for the smooth structure it is Theorem 18.11. Glöckner and Neeb do not treat the sphere; we do not claim this construction is new. □

18.3 The unitary group of a Banach $*$-algebra

Let A be a real Banach algebra with a continuous involution $a \mapsto a^*$ (in Lean: a complete normed $\mathbb{R}$-algebra which is a $*$-ring and a $*$-module, with continuous $*$). For instance $A = B(H)$, the bounded operators on a complex Hilbert space H, seen as a real algebra. The unitary group is

$$U(A) = \{u \in A : u^*u = 1 = uu^*\},$$

and its model is the space $\mathfrak{u}(A) = \{X \in A : X^* = -X\}$ of skew-adjoint elements.

Remark 18.13 (Completeness). Here A is complete, so A is a Banach manifold and $U(A)$ is a Banach submanifold of it. Completeness is used twice: the invertible elements form an open set, and inversion is smooth on it. Both facts may fail in an incomplete normed algebra.

The slice charts are Cayley transforms. On the open set $\{a : a + 1 \text{ invertible}\}$ we let

$$\varphi(a) = 1 - 2(a+1)^{-1} = (a-1)(a+1)^{-1}, \qquad \varphi^{-1}(X) = 2(1-X)^{-1} - 1 = (1+X)(1-X)^{-1}.$$

At other points $p \in U(A)$ we compose with the left translation $a \mapsto p^*a$, a linear homeomorphism of A with inverse $a \mapsto pa$.

Lemma 18.14 (The Cayley transform)**.** *The Cayley transform φ is a smooth chart of A with smooth inverse. If $a + 1$ is invertible, then $a \in U(A)$ if and only if $\varphi(a) \in \mathfrak{u}(A)$. For every $p \in U(A)$, the chart $a \mapsto \varphi(p^*a)$ is a slice chart for $U(A)$ around p.*

Proof. The inverse of φ is checked by a direct computation with units, and smoothness follows from the smoothness of inversion on the open set of units. The equivalence is pure algebra: writing $s = a + 1$ and using $(s^{-1})^* = (s^*)^{-1}$, the condition $\varphi(a)^* = -\varphi(a)$ unwinds to $a^*a = 1 = aa^*$; no spectral theory is needed. Finally $p^*p + 1 = 2$ is invertible, so p lies in the domain of the translated chart. □

Theorem 18.15 (The unitary group is a split submanifold)**.** *The unitary group $U(A)$ is a split submanifold of A modeled on $\mathfrak{u}(A)$, and*

$$T_pU(A) = \{w \in A : p^*w \in \mathfrak{u}(A)\} = p\,\mathfrak{u}(A).$$

In particular this holds for the unitary group of $B(H)$, H a complex Hilbert space.

Proof. The skew-adjoint elements are closed and complemented by the self-adjoint ones, with projection $X \mapsto (X - X^*)/2$. The derivative of φ at 1 is $w \mapsto w/2$, so the derivative of the translated chart at p is $w \mapsto p^*w/2$, and the tangent space is read off as in Theorem 18.9. □

The natural projector onto $T_pU(A)$ is $E_pw = \frac{1}{2}(w - pw^*p)$; writing $X = p^*w$, it is $E_pw = p\,\frac{1}{2}(X - X^*)$, the skew-adjoint part translated back to p.

Theorem 18.16 (Geodesics of the unitary group)**.** *The maps $E_p w = \frac{1}{2}(w - pw^*p)$, $p \in U(A)$, form a field of projectors onto the tangent spaces of $U(A)$. A curve α in $U(A)$ whose ambient curve is smooth is a geodesic of the associated connection if and only if*

$$\ddot\alpha(t) = \alpha(t)\,\ddot\alpha(t)^*\,\alpha(t) \quad \text{for all } t.$$

For $p \in U(A)$ and $X \in \mathfrak{u}(A)$, the curve $t \mapsto p\,e^{tX}$ is a geodesic.

Proof. Using $p^*p = 1 = pp^*$ one checks $p(E_p w)^* p = \frac{1}{2}(pw^*p - w)$, whence $E_p^2 = E_p$, and $p^*E_p w = \frac{1}{2}(p^*w - w^*p)$ is skew-adjoint. The geodesic equation is the general criterion $E_\alpha \ddot\alpha = 0$. For $\alpha(t) = pe^{tX}$, e^{tX} is unitary and commutes with X, and $\ddot\alpha = pe^{tX}X^2$; a short computation gives $\alpha\ddot\alpha^*\alpha = pe^{tX}X^2$. □

Remark 18.17. The structure of Lie group of $U(A)$ (smoothness of the product and of the inversion in these charts) is not addressed in this chapter; we only use $U(A)$ as a submanifold of A. The connection of Theorem 18.16 is defined directly from the field of projectors, with no metric behind it. The header of the Lean file also mentions $B(H)$ for a real Hilbert space H; the formalized statement `isSplitSubmanifold_unitary_clm` covers complex Hilbert spaces.

18.4 Smooth periodic functions and $\mathrm{Diff}^+(S^1)$

18.4.1 The Fréchet space C^∞_{per}

Let C^∞_{per} be the space of smooth 1-periodic functions $u\colon \mathbb{R} \to \mathbb{R}$ (smooth functions on the circle $S^1 = \mathbb{R}/\mathbb{Z}$), with the topology of uniform convergence of all the derivatives, that is, the topology of the seminorms

$$p_k(u) = \sup_{x \in \mathbb{R}} |u^{(k)}(x)|, \qquad k \in \mathbb{N}.$$

Definition 18.18 (C^∞_{per})**.** The topology and the uniform structure of C^∞_{per} are those induced by the injective linear map

$$J\colon C^\infty_{per} \to \prod_{k \in \mathbb{N}} BC(\mathbb{R}), \qquad J(u) = (u, u', u'', \dots),$$

into a countable product of Banach spaces of bounded continuous functions. The seminorm p_k is the sup norm of the k-th component of J.

All the structure is inherited from the product: C^∞_{per} is a Hausdorff locally convex topological vector space, and it is metrizable, since the product is countable. In Lean these are instances on `CInfPer`.

Proposition 18.19 (Convergence and seminorms)**.** *A net u_i converges to u in C^∞_{per} if and only if $u_i^{(k)} \to u^{(k)}$ uniformly on $\mathbb{R}$ for every k. Moreover $p_k(u) \le C$ if and only if $|u^{(k)}(x)| \le C$ for all x.*

Proof. J is a uniform embedding and convergence in a product is convergence of every component. □

Theorem 18.20 (C^∞_{per} is a Fréchet space)**.** *The range of J is closed. Consequently C^∞_{per} is complete, and it is a Fréchet space: locally convex, Hausdorff, metrizable and complete.*

Proof. If $u_n^{(k)} \to g_k$ uniformly for every k, the theorem on uniform limits of derivatives shows that g_0 is differentiable with $g_0' = g_1$, and inductively $g_0^{(k)} = g_k$; so g_0 is smooth, periodic, and $J(g_0) = (g_k)_k$. A closed subspace of the complete space $\prod_k BC(\mathbb{R})$ is complete, and J is a uniform embedding. □

Remark 18.21. The space C^∞_{per} is not normable, so it is a Fréchet space that is not a Banach space. Non-normability is the standard fact that a normable Fréchet space has a bounded neighbourhood of 0; it is not formalized here. Completeness, which we do prove, is what makes the manifolds modeled on C^∞_{per} below Fréchet manifolds in the strict sense.

18.4.2 The universal cover of $\mathrm{Diff}^+(S^1)$

Let $\pi\colon \mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z}$ be the projection. We recall the classical facts that relate circle maps to their lifts; they are *not* formalized, and the formalization works directly with the lifts.

Remark 18.22 (From the circle to its lifts). The map π is a covering map and a local diffeomorphism.

- *Existence and uniqueness of lifts.* Since $\mathbb{R}$ is simply connected, every continuous $f\colon S^1 \to S^1$ has a continuous lift $\varphi\colon \mathbb{R} \to \mathbb{R}$, $\pi \circ \varphi = f \circ \pi$, by the lifting criterion [Hat02, §1.3, Prop. 1.33] (see also the homotopy lifting property, [Hat02, Prop. 1.30]), and two lifts differ by an integer constant: integer translations are deck transformations, and a lift is determined by its value at one point [Hat02, Prop. 1.34].
- *Degree.* $\varphi(x+1) - \varphi(x)$ is a continuous integer valued function, hence a constant d, the degree of f [KH95, Lemma 2.4.3, Def. 2.4.4]. A homeomorphism has degree ± 1 (the degree is multiplicative, [KH95, Prop. 8.2.2]), and it preserves the orientation exactly when $d = 1$ [KH95, §11.1, p. 388], i.e. $\varphi(x+1) = \varphi(x) + 1$; then $\varphi = \mathrm{id} + u$ with u 1-periodic.
- *Smoothness.* Since π is a local diffeomorphism, φ is smooth if and only if f is, and φ is a diffeomorphism of $\mathbb{R}$ exactly when f is a diffeomorphism of S^1 (the inverse of f lifts to the inverse of φ).

So $\mathrm{Diff}^+(S^1) \cong \Omega/\mathbb{Z}$, where $n \in \mathbb{Z}$ acts by $u \mapsto u + n$ and Ω is the set defined below. This is the description of Herman [Her79, Ch. I, 1.7], where $D^r(T^1) = \{\mathrm{id} + \varphi\}$ is called the universal cover of $\mathrm{Diff}^r_+(T^1)$; Hamilton [Ham82, Part I, Example 4.2.6] parametrizes $\mathrm{Diff}^+(S^1)$ instead by $S^1 \times \{f' > 0,\ \int f' = 2\pi\}$. Since Ω is convex, hence contractible, $\Omega \to \mathrm{Diff}^+(S^1)$ is the universal cover $\widetilde{\mathrm{Diff}}^+(S^1)$ [Her79, Introduction (Remarque importante)]. For Lie groups of diffeomorphisms in the locally convex setting see [GN26]. In Mathlib, the monotone lifts with $\varphi(x+1) = \varphi(x) + 1$ of circle maps appear as `CircleDeg1Lift`.

An orientation preserving diffeomorphism of the circle thus lifts to a diffeomorphism $\varphi = \mathrm{id} + u$ of $\mathbb{R}$ with $u \in C^\infty_{per}$ and $\varphi' = 1 + u' > 0$.

Definition 18.23 (The domain Ω)**.** We let

$$\Omega = \{u \in C^\infty_{per} : 1 + u'(x) > 0 \text{ for all } x \in \mathbb{R}\},$$

and for $u \in C^\infty_{per}$ we let $\varphi_u = \mathrm{id} + u$ be its lift.

Proposition 18.24 (Ω is open and convex)**.** *The set Ω is open and convex in C^∞_{per}, and contains 0. Hence Ω is a manifold modeled on C^∞_{per}; since the model is complete, it is a Fréchet manifold. It is not a Banach manifold, since the model is not normable (an unformalized fact).*

Proof. For $u \in \Omega$, the continuous periodic function $1 + u'$ attains a positive minimum c on the compact circle. If $p_1(v - u) < c$, then $1 + v' > 0$, so Ω contains a neighbourhood of u. Convexity is the identity $1 + (au + bv)' = a(1 + u') + b(1 + v')$ for $a, b \geq 0$, $a + b = 1$. An open subset of a locally convex space is a manifold with the restricted identity chart. □

Proposition 18.25 (Lifts are diffeomorphisms)**.** *For $u \in \Omega$, the lift φ_u is an increasing homeomorphism of $\mathbb{R}$, smooth with smooth inverse, and both φ_u and φ_u^{-1} commute with the translation by 1.*

Proof. $\varphi_u' = 1 + u' > 0$ gives strict monotonicity. Since u is bounded, $\varphi_u(x) \to \pm\infty$ as $x \to \pm\infty$, so φ_u is surjective by the intermediate value theorem. The inverse is smooth by the one-variable inverse function theorem, its derivative being $1/\varphi_u' \circ \varphi_u^{-1}$. □

Theorem 18.26 (Ω is closed under the group operations)**.** *Let $u, w \in \Omega$. Then $\varphi_u^{-1} = \mathrm{id} + v$ with $v \in \Omega$, and*

$$\varphi_u \circ \varphi_w = \mathrm{id} + (w + u \circ \varphi_w), \qquad w + u \circ \varphi_w \in \Omega.$$

Proof. The function $v = \varphi_u^{-1} - \mathrm{id}$ is smooth and periodic by Proposition 18.25, and $(\varphi_u^{-1})'(y) = 1/\varphi_u'(\varphi_u^{-1} y) > 0$. For the composition, $w + u \circ \varphi_w$ is smooth and periodic since φ_w commutes with the translation by 1, and by the chain rule $1 + (w + u \circ \varphi_w)' = (1 + u' \circ \varphi_w)(1 + w') > 0$. □

Conversely, Ω captures all the diffeomorphisms of $\mathbb{R}$ that commute with the translation by 1, and positivity of the derivative does not need to be assumed.

Theorem 18.27 (Characterization of Ω)**.** *For $u \in C^\infty_{per}$, $u \in \Omega$ if and only if φ_u has a differentiable left inverse. Consequently, if $\varphi\colon \mathbb{R} \to \mathbb{R}$ is smooth, has a differentiable left inverse and satisfies $\varphi(x + 1) = \varphi(x) + 1$, then $\varphi = \mathrm{id} + u$ for a unique $u \in \Omega$.*

Proof. A continuous injective map of $\mathbb{R}$ is strictly monotone, and it cannot be decreasing since it commutes with the translation by 1; so $\varphi_u' \geq 0$, and the chain rule $\psi'(\varphi_u x)\, \varphi_u'(x) = 1$ excludes zeros. For the second claim take $u = \varphi - \mathrm{id}$, which is smooth and periodic; uniqueness is the injectivity of $u \mapsto \mathrm{id} + u$. □

So Ω is exactly the group of diffeomorphisms of $\mathbb{R}$ commuting with the translation by 1, that is (by Remark 18.22), the universal cover of $\mathrm{Diff}^+(S^1)$. Lifts of orientation reversing diffeomorphisms satisfy $\varphi(x + 1) = \varphi(x) - 1$ instead, and they are excluded by the hypothesis.

Remark 18.28 (What is not proved). Theorem 18.26 shows that Ω is closed under composition and inversion, so it is a group, and Proposition 18.24 shows that it is a Fréchet manifold. The smoothness of the maps $\Omega \times \Omega \to \Omega$, $(u, w) \mapsto w + u \circ \varphi_w$, and $\Omega \to \Omega$, $u \mapsto v$, is *not* proved here; this is what would make Ω a Lie group, and it belongs to the Lie group phase of the project. We remark that no general inverse function theorem is available in Fréchet spaces, so the smoothness of the inversion has to be obtained by a direct argument.

Chapter 19

Counterexamples

Counterexamples that separate notions which agree in finite dimensions. This chapter collects those about immersions and Finsler structures: a weak immersion that is not an immersion ($\ell^1 \to \ell^2$), an injective immersion that is not initial (the figure eight), an immersion between Banach spaces that is not split ($c_0 \subseteq \ell^\infty$, by Phillips' lemma), and a Finsler length structure that is not a Finsler metric. Three more are stated next to the results they limit: McAlpin's complete Riemannian manifold without shortest curves (Example 12.13), a bounded Finsler structure on an incomplete model whose distance does not give the manifold topology (Theorem 13.51), and a weak Riemannian metric without Levi-Civita connection (Theorem 13.37).

19.1 Weak immersions, injective immersions and split immersions

The notions of immersion form a chain: split immersions are immersions, immersions are weak immersions, and injective immersions need not be initial. In finite dimensions the first two notions coincide. The three examples of this section show that every implication is strict, and that the first two separations occur already between Banach spaces.

19.1.1 A weak immersion that is not an immersion

Let $\iota\colon \ell^1 \to \ell^2$ be the inclusion of real sequences. Both spaces are Banach spaces, hence manifolds with a single chart.

Proposition 19.1 ($\ell^1 \to \ell^2$)**.** *The inclusion ι is continuous, linear and injective, hence a weak immersion. It is not bounded below, hence it is not an immersion.*

Proof. Continuity follows from the closed graph theorem: the graph of ι is closed because the coordinate evaluations are continuous on both spaces. The differential of ι at every point is ι, which is injective. For $x_n = e_0 + \cdots + e_{n-1}$ we have $\|x_n\|_1 = n$ and $\|x_n\|_2 = \sqrt{n}$, so no inequality $m\|x\|_1 \le \|\iota x\|_2$ with $m > 0$ can hold, and ι is not a topological isomorphism onto its range. □

Remark 19.2. The range of ι is dense in ℓ^2 and not closed. In finite dimensions an injective linear map is automatically a topological isomorphism onto a closed subspace; in infinite dimensions the condition that the differential be an embedding with closed range is a genuine additional requirement, and this is the reason to distinguish weak immersions from immersions.

19.1.2 An injective immersion that is not initial: the figure eight

Let $\theta(t) = 2\arctan t \in (-\pi, \pi)$ and

$$j\colon \mathbb{R} \to \mathbb{R}^2, \qquad j(t) = (\sin\theta(t), \sin 2\theta(t)).$$

As t runs over $\mathbb{R}$, $j(t)$ traces the figure eight (a lemniscate), passing through the origin at $t = 0$ and tending to the origin again as $t \to \pm\infty$.

Proposition 19.3 (The figure eight is immersed)**.** *The map j is smooth, injective and has nonvanishing velocity; hence it is an injective immersion.*

Proof. On $(-\pi, \pi)$ the pair $(\sin a, \sin 2a)$ determines a, which gives injectivity. If $\dot{j}(t) = 0$, then $\cos\theta = 0$ and $\cos 2\theta = 0$ at the same time, which is impossible. A nonzero vector spans a closed line, so $v \mapsto v\,\dot{j}(t)$ is a topological isomorphism onto a closed subspace. □

Theorem 19.4 (The figure eight is not initial)**.** *Let $\sigma(s) = \frac{1}{2}\arctan s$ and $f(s) = (\sin\sigma, -\sin 2\sigma)$. Then f is a smooth curve in $\mathbb{R}^2$ with values in $j(\mathbb{R})$, but its unique lift g with $j \circ g = f$ is not continuous at 0. Hence j is not an initial submanifold.*

Proof. The curve f crosses the origin along the branch that j reaches only as $t \to \pm\infty$. Explicitly, $g = \tan(\lambda/2)$ with $\lambda(s) = \pi - \sigma(s)$ for $s > 0$, $\lambda(s) = -\pi - \sigma(s)$ for $s < 0$ and $\lambda(0) = 0$; trigonometric identities give $j \circ g = f$. For $s > 0$ we have $g(s) > 1$, while $g(0) = 0$. If j were initial, g would be smooth, in particular continuous. □

Remark 19.5. This is the classical finite dimensional example; it shows that the condition defining initial submanifolds is not a consequence of being an injective immersion, and has nothing to do with infinite dimensions. The topology of $\mathbb{R}$ transported by j is strictly finer than the topology induced by $\mathbb{R}^2$ on the figure eight.

19.1.3 An immersion that is not split: $c_0 \subseteq \ell^\infty$

We realize ℓ^∞ as the Banach space of bounded real sequences and c_0 as the closed subspace of sequences tending to 0. The classical obstruction is the following result of Phillips [Phi40, 7.5, pp. 539–540], stated there for the space c of convergent sequences, which contains c_0 with codimension one; we formalize the short proof of Whitley [Whi66].

Theorem 19.6 (Phillips' lemma)**.** *There is no continuous linear projection of ℓ^∞ onto c_0.*

The proof rests on an uncountable family of almost disjoint subsets of $\mathbb{N}$, and on the fact that a functional vanishing on c_0 can only see countably many of them.

Lemma 19.7 (An almost disjoint family)**.** *For each $A \subseteq \mathbb{N}$ there is an infinite set $N_A \subseteq \mathbb{N}$ such that $N_A \cap N_B$ is finite whenever $A \neq B$. The indicator $\chi_A \in \ell^\infty$ of N_A does not belong to c_0.*

Proof. Index the finite binary words by $\mathbb{N}$, and let N_A be the set of indices of the initial segments of the indicator function of A. If A and B differ at i_0, their initial segments of length larger than i_0 are different, so $N_A \cap N_B$ is finite. Since N_A is infinite, χ_A takes the value 1 infinitely often. □

Lemma 19.8 (Whitley)**.** *Let f be a continuous linear functional on ℓ^∞ that vanishes on c_0. If $|x_k| \leq 1$ for all but finitely many k, then $|f(x)| \leq \|f\|$. Consequently, $f(\chi_A) \neq 0$ only for countably many $A \subseteq \mathbb{N}$.*

Proof. For the first claim, modify x on a finite set to obtain y with $\|y\| \le 1$; then $x - y \in c_0$ and $f(x) = f(y)$. For the second, suppose $|f(\chi_A)| > 1/m$ for every A in a finite set $\mathcal{F}$, and let $x = \sum_{A\in\mathcal{F}} \pm\chi_A$, with the signs of $f(\chi_A)$. Off the finite set where two of the N_A meet, $|x_k| \le 1$; hence $\#\mathcal{F}/m < f(x) \le \|f\|$. So each set $\{A : |f(\chi_A)| > 1/m\}$ is finite. □

Proof of Theorem 19.6. Suppose P is a continuous projection onto c_0, and apply Lemma 19.8 to the functionals $f_n = e_n \circ (1 - P)$, where e_n is the n-th coordinate. The subsets of $\mathbb{N}$ are uncountable, so some A avoids the countable union of the exceptional sets. Then $(1-P)\chi_A = 0$, that is, $\chi_A = P\chi_A \in c_0$, a contradiction. □

Corollary 19.9 (Embedded but not split)**.** *The subspace c_0 is an embedded submanifold of ℓ^∞ that is not split, and the inclusion $c_0 \to \ell^\infty$ is an immersion that is not a split immersion.*

Proof. Combine Propositions 18.1 and 18.2 with Theorem 19.6. □

Remark 19.10. In Hilbert spaces every closed subspace is complemented, so the distinction between embedded and split submanifolds, and between immersions and split immersions, is invisible there. For Banach spaces it is not, and Corollary 19.9 gives the standard witness.

19.2 A Finsler length structure that is not a Finsler metric

The difference between the two notions is not an infinite-dimensional phenomenon. On $\mathbb{R}^2$ with the sup norm, let

$$\beta(x,y) = \frac{\max(0, x^2 - |y|)}{x^2 + |y|} \quad (\beta(0,0) = 0), \qquad \mu_p(v) = \|v\| + \beta(p)\,|v_2|.$$

The weight β takes values in $[0,1]$, is supported in the cusp $\{|y| < x^2\}$, equals 1 on the x-axis minus the origin, and is continuous away from the origin.

Proposition 19.11. *Each μ_p is a reversible Finsler norm.*

Proof. It is the sum of the norm and a nonnegative multiple of the seminorm $|v_2|$. □

Theorem 19.12. *The field μ is a Finsler length structure but not a Finsler metric.*

Proof. It is not continuous on $T\mathbb{R}^2$ at $(0, e_2)$: $\mu_{(t,0)}(e_2) = 2$ for $t \neq 0$, while $\mu_0(e_2) = 1$. Let now γ be a C^1 curve. Away from the times where γ passes through the origin, $\beta \circ \gamma$ is continuous. Suppose that $\gamma(t_0) = 0$. If $\dot\gamma_2(t_0) \neq 0$, then $|\gamma_2(t)| \ge c\,|t - t_0|$ while $\gamma_1(t)^2 = O(|t-t_0|^2)$, so near t_0 the curve stays outside the cusp and $\beta(\gamma(t)) = 0$. If $\dot\gamma_2(t_0) = 0$, then $0 \le \beta(\gamma)\,|\dot\gamma_2| \le |\dot\gamma_2| \to 0$. □

The example shows what a C^1 curve can and cannot see: through a point p it only meets the pairs (p, v) that are compatible with the way it approaches p. The pairs $((1/n, 0), e_2)$, which converge to $(0, e_2)$ in $T\mathbb{R}^2$, are not of this kind.

Bibliography

[AI89] T. Ya. Azizov and I. S. Iokhvidov, *Linear Operators in Spaces with an Indefinite Metric*, Wiley, 1989.

[Atk75] C. J. Atkin, *The Hopf–Rinow theorem is false in infinite dimensions*, Bull. London Math. Soc. **7** (1975), 261–266.

[BBI01] D. Burago, Y. Burago and S. Ivanov, *A Course in Metric Geometry*, Graduate Studies in Mathematics **33**, American Mathematical Society, 2001.

[Bel05] D. Beltiță, *Integrability of analytic almost complex structures on Banach manifolds*, Ann. Global Anal. Geom. **28** (2005), 59–73.

[BL23] D. Beltiţă and G. Larotonda, *Unitary group orbits versus groupoid orbits of normal operators*, J. Geom. Anal. **33** (2023), no. 3, Paper No. 95, doi:10.1007/s12220-022-01187-5.

[Bog74] J. Bognár, *Indefinite Inner Product Spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete **78**, Springer, 1974.

[Car57] H. Cartan, *Variétés analytiques réelles et variétés analytiques complexes*, Bull. Soc. Math. France **85** (1957), 77–99, doi:10.24033/bsmf.1481.

[dMU21] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in *Automated Deduction – CADE 28*, Lecture Notes in Comput. Sci. **12699**, Springer, 2021, 625–635.

[Eke78] I. Ekeland, *The Hopf–Rinow theorem in infinite dimension*, J. Differential Geom. **13** (1978), 287–301.

[GN26] H. Glöckner and K.-H. Neeb, *Infinite-Dimensional Lie Groups*, book manuscript, arXiv:2602.12362, 2026.

[GLT] T. Goliński, G. Larotonda, A. B. Tumpach, *Nijenhuis operators on Banach homogeneous spaces*, Rend. Lincei Mat. Appl. **35** (2024), 713–739, doi:10.4171/RLM/1057.

[Gra58] H. Grauert, *On Levi's problem and the imbedding of real-analytic manifolds*, Ann. of Math. (2) **68** (1958), 460–472, doi:10.2307/1970257.

[Ham82] R. S. Hamilton, *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. **7** (1982), 65–222.

[Hat02] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.

[Her79] M. R. Herman, *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations*, Publ. Math. IHÉS **49** (1979), 5–233.

[Hir76] M. W. Hirsch, *Differential Topology*, Graduate Texts in Mathematics **33**, Springer, 1976.

[Jor82] H. Joris, *Une C^∞-application non-immersive qui possède la propriété universelle des immersions*, Arch. Math. **39** (1982).

[KH95] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Encyclopedia of Mathematics and its Applications **54**, Cambridge University Press, 1995.

[KMW24] Y. Karshon, D. Miyamoto and J. Watts, *Diffeological submanifolds and their friends*, Differential Geom. Appl. **96** (2024), Paper No. 102170, doi:10.1016/j.difgeo.2024.102170.

[Lan99] S. Lang, *Fundamentals of Differential Geometry*, Graduate Texts in Mathematics **191**, Springer, 1999.

[Lar19] G. Larotonda, *The metric geometry of infinite dimensional Lie groups and their homogeneous spaces*, Forum Math. **31** (2019), 1567–1605.

[Lar26] G. Larotonda, *Estructuras geométricas para las variedades de Banach*, 2nd ed., unpublished book manuscript, version 3 of 15 August 2026, https://mate.dm.uba.ar/~glaroton/Estructuras_v3.pdf.

[LCA] @glaroton, *lc-analysis: Michal–Bastiani calculus on locally convex spaces*, Lean library, https://github.com/glaroton/lc-analysis.

[Lem98] L. Lempert, *The Dolbeault complex in infinite dimensions I*, J. Amer. Math. Soc. **11** (1998), 485–520.

[Mai63] B. Maissen, *Über Topologien im Endomorphismenraum eines topologischen Vektorraumes*, Math. Ann. **151** (1963), 283–285.

[Mar72] J. E. Marsden, *Darboux's theorem fails for weak symplectic forms*, Proc. Amer. Math. Soc. **32** (1972), 590–592, doi:10.1090/S0002-9939-1972-0293678-X.

[Men13] A. C. G. Mennucci, On asymmetric distances, *Anal. Geom. Metr. Spaces* **1** (2013), 200–231.

[mC20] The mathlib Community, *The Lean mathematical library*, in *Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020)*, 2020, 367–381.

[Mos65] J. Moser, *On the volume elements on a manifold*, Trans. Amer. Math. Soc. **120** (1965), 286–294.

[MM05] P. W. Michor and D. Mumford, *Vanishing geodesic distance on spaces of submanifolds and diffeomorphisms*, Doc. Math. **10** (2005), 217–245.

[Nee06] K.-H. Neeb, *Towards a Lie theory of locally convex groups*, Jpn. J. Math. **1** (2006), 291–468.

[NN57] A. Newlander, L. Nirenberg, *Complex analytic coordinates in almost complex manifolds*, Ann. of Math. **65** (1957), 391–404.

[Pal66] R. S. Palais, *Lusternik–Schnirelman theory on Banach manifolds*, Topology **5** (1966), 115–132.

[Pen70] J.-P. Penot, *Sur le théorème de Frobenius*, Bull. Soc. Math. France **98** (1970), 47–80.

[Pat99] I. Patyi, *On the $\bar{\partial}$-equation in a Banach space*, Bull. Soc. Math. France **128** (2000), 391–406.

[Phi40] R. S. Phillips, *On linear transformations*, Trans. Amer. Math. Soc. **48** (1940), 516–541.

[Sch23] A. Schmeding, *An Introduction to Infinite-Dimensional Differential Geometry*, Cambridge Studies in Advanced Mathematics **202**, Cambridge University Press, 2023, doi:10.1017/9781009091251.

[Wei69] A. Weinstein, *Symplectic structures on Banach manifolds*, Bull. Amer. Math. Soc. **75** (1969), 1040–1041, doi:10.1090/S0002-9904-1969-12353-0.

[Wei71] A. Weinstein, *Symplectic manifolds and their Lagrangian submanifolds*, Adv. Math. **6** (1971), 329–346.

[Whi66] R. Whitley, *Projecting m onto c_0*, Amer. Math. Monthly **73** (1966), 285–286.