Michal–Bastiani calculus on locally convex spaces

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Chapter 1

Introduction

This blueprint describes the mathematics formalized in the Lean library lc-analysis: the differential and integral calculus of Michal and Bastiani on locally convex spaces. Every definition and theorem below is linked to its Lean counterpart, and the dependency graph shows how the results build on each other. All the statements in this document are proved in Lean, with no `sorry`, and the main theorems depend only on the standard axioms of Lean.

Why Michal–Bastiani calculus

Infinite-dimensional Lie groups and manifolds of mappings are modeled on spaces that are not normable: Fréchet spaces of smooth functions such as $C^\infty(M)$, spaces of test functions, direct limits. On such spaces the Fréchet derivative is not available, since there is no norm with which to measure the remainder. One is faced with the problem of choosing a notion of differentiability that does not depend on a norm, and that still has a chain rule, a mean value theorem and a fundamental theorem of calculus.

The answer that has become standard in infinite-dimensional Lie theory goes back to Michal (1938, [Mic38]) and Bastiani (1964, [Bas64]). It was developed further by Keller [Kel74], Hamilton [Ham82], Milnor [Mil84], Glöckner [Glo02] and Neeb [Nee06], and its main reference today is the book of Glöckner and Neeb [GN26], which we follow closely; [GN26, 1.2.13] refers to its numbered result 1.2.13. We remark that this is not the convenient setting of Kriegl and Michor [KM97]: smoothness is defined here through the topology of the model space, and not through smooth curves.

The idea is to ask for less than in the Banach case. A map is C^1 if all its directional derivatives exist and depend continuously on the point *and* on the direction, jointly. No uniformity in the direction is required. In normed spaces this notion is weaker than Fréchet differentiability, but the two give the same smooth maps (Chapter 8).

Mathlib [mC20], the mathematical library of the Lean proof assistant [dMU21], has a complete Fréchet calculus on normed spaces, and a theory of topological vector spaces and locally convex spaces. It does not have a differential calculus beyond the normed case. The purpose of lc-analysis is to fill this gap, as a foundation for the formalization of infinite-dimensional Lie theory.

Standing assumptions

Unless otherwise stated, E, F and G denote real locally convex Hausdorff topological vector spaces, $U \subseteq E$ is open, and $f\colon U \to F$. These spaces are not assumed to be complete in any sense; whenever some form of completeness is needed (mainly for the existence of integrals) it appears as an explicit hypothesis. The Lean statements sometimes need less than the standing assumptions: for instance, the definition of weak integrals makes sense in any topological vector space. The precise hypotheses can be read in the linked Lean declarations.

Two tools replace norm estimates throughout:

- **Hahn–Banach.** Continuous linear functionals separate points from closed convex sets. In particular, a vector is determined by its images under all continuous linear functionals.
- **Local convexity.** The closed convex neighborhoods of 0 form a basis of the topology.

Most proofs reduce a statement in F to statements about real functions, by composing with continuous linear functionals, and then the one-variable calculus of Mathlib applies. Where in the normed case a norm estimate would bound an increment, here we show instead that the increment lies in a prescribed closed convex neighborhood.

Contents

The chapters follow the folders of the library, so that each part can be read on its own.

- **Calculus.** Chapter 2 defines C^k maps and proves the first-order theory: linearity of the differential, the mean value theorem, the chain rule, partial derivatives and the symmetry of the second differential. Chapter 3 compares the recursive definition of C^k with Bastiani's definition through iterated directional derivatives, and collects the rules of calculus (bilinear maps, Taylor's formula, the second-order chain rule).
- **Integration.** Chapter 4 defines weak integrals, and Chapter 5 studies the completeness hypotheses under which they exist, including the equivalence of two definitions of Mackey completeness.
- **Mapping spaces.** Chapter 6 gives spaces of C^k maps the compact-open C^k topology and proves the exponential law; Chapter 7 extends the calculus to locally convex subsets with dense interior, the domains of charts of manifolds with boundary.
- **Normed spaces.** Chapter 8 compares Michal–Bastiani and Fréchet differentiability on normed spaces, and treats maps with bounded below derivative.
- **Analytic maps.** Chapter 9 develops complex and real analytic maps on locally convex spaces.
- **Differential equations and integrability.** Chapter 10 treats flows on Banach spaces and equations with locally convex parameters, Chapter 11 the Frobenius theorem, and Chapter 12 complex structures and the Newlander–Nirenberg theorem.
- **Spaces.** Chapter 13 treats direct limits of topological spaces and k_ω-spaces, Chapter 14 locally convex direct sums, direct limits and Silva spaces, and Chapter 15 Krein spaces.
- **Limits of the theory.** Chapter 16 collects the formalized counterexamples beyond the normed setting, and Chapter 17 explains the design choices they impose.

What is new and what is not

The mathematics is classical: every result in this document appears in the cited literature, and the proofs follow it. The few places where the argument differs from the sources are pointed out in remarks. As far as we know, Michal–Bastiani calculus had not been formalized before, neither in Lean nor in another proof assistant.

How the library was written

@glaroton set the mathematical program, chose the framework and the references, fixed the design constraints and reviewed the results. The Lean proofs were written by the AI assistant Claude (Anthropic), working under his direction. Correctness does not rest on trusting either of them: Lean's kernel checks every proof, and the statements are kept close to the literature so that they can be compared with it directly.

Chapter 2

Michal–Bastiani differentiability

On a Banach space a map is C^1 when its derivative is a continuous map into the space of bounded operators, with the operator norm. On a general locally convex space E there is no canonical topology on the space of continuous linear maps $E \to F$ that would play this role, and the definition has to be rephrased. The solution of Michal [Mic38] and Bastiani [Bas64] is to consider the derivative as a function of two variables, the point and the direction, and to ask for joint continuity. In the recursive form used by Glöckner, which we adopt, C^{k+1} means that this function of two variables is C^k.

In this chapter we define C^k maps between locally convex spaces and prove the basic results of first-order calculus. After the definitions, we prove the mean value theorem, which in this setting takes the form of an inclusion in a closed convex set. From it we obtain Glöckner's lemma on the continuous extension of difference quotients, and from this lemma the linearity of the differential and the chain rule. We close the chapter with partial derivatives, the symmetry of the second differential, and the comparison with Fréchet C^k maps on normed spaces. The references are [GN26, §1.2] and [Nee06, §I.2].

2.1 Directional derivatives and C^k maps

Definition 2.1 (Directional derivative)**.** Let $f\colon E \to F$ and $x, h \in E$. We say that f has derivative $y \in F$ at x in the direction h if

$$\lim_{t \to 0,\ t \neq 0} \frac{1}{t}(f(x+th) - f(x)) = y.$$

We write $\mathrm{d}f(x)(h)$ for this limit when it exists, and regard $\mathrm{d}f$ as a function on $E \times E$.

The limit is taken in F, which is Hausdorff, so it is unique when it exists. Note that $\mathrm{d}f(x)(0) = 0$ always, and that no linearity in h is built into the definition.

Remark 2.2. In Lean, $\mathrm{d}f$ is a total function: it takes an arbitrary value where the limit does not exist. This is harmless, since it is only used where the limit is known to exist.

Definition 2.3 (C^k and smooth maps)**.** Let $U \subseteq E$ be open. A map f is C^0 on U if it is continuous on U. It is C^{k+1} on U if it is continuous on U, all directional derivatives $\mathrm{d}f(x)(h)$ exist for $x \in U$, $h \in E$, and $\mathrm{d}f\colon U \times E \to F$ is C^k on the open set $U \times E$. The map f is smooth on U if it is C^k for every k.

Thus f is C^1 when its directional derivatives exist and are jointly continuous in the point and the direction. This is [GN26, 1.2.1 and 1.3.1], in the recursive form used by Glöckner; Bastiani's original definition, through the continuity of all the iterated differentials, is equivalent (Theorem 3.5).

Remark 2.4. Two features of the definition deserve a comment. First, linearity of $h \mapsto \mathrm{d}f(x)(h)$ is not assumed; it is a theorem (Theorem 2.12), and it depends on the joint continuity in an essential way (Example 2.13 below). Second, no topology on the space of continuous linear maps $E \to F$ is involved: the regularity of $\mathrm{d}f$ is measured on the locally convex space $E \times E$, so that the induction stays within the category of locally convex spaces.

Remark 2.5. The recursive definition quantifies over the domain of $\mathrm{d}f$, which is $U \times E$ rather than U. As a consequence, in Lean the domains of C^k maps live in a fixed universe. Codomains are unrestricted.

We record the two basic examples, which are used throughout.

Lemma 2.6 (Continuous linear maps are smooth)**.** *Every continuous linear map $L\colon E \to F$ is smooth, and $\mathrm{d}L(x)(h) = Lh$.*

Proof. The difference quotients of L are constant, equal to Lh. Then $\mathrm{d}L = L \circ \pi_2$ is again continuous and linear, and the claim follows by induction on k. □

Lemma 2.7 (Pairs)**.** *If $f\colon U \to F$ and $g\colon U \to G$ are C^k, so is $x \mapsto (f(x), g(x))$.*

Proof. Directional derivatives of a pair are computed componentwise, so $\mathrm{d}(f, g)$ is the pair $(\mathrm{d}f, \mathrm{d}g)$; the claim follows by induction on k. □

2.2 The mean value theorem

The classical mean value inequality $\|f(x+h) - f(x)\| \le \sup_s \|Df(x+sh)\|\,\|h\|$ has no meaning without a norm. Its replacement is an inclusion: increments of f lie in the closed convex hull of the derivatives along the segment. Since closed convex neighborhoods form a basis of the topology, this inclusion carries the same information as a family of estimates, one for each continuous seminorm.

Lemma 2.8 (Mean value inclusion)**.** *Let $z, w \in E$ and suppose that f has derivative $D(s)$ at $z + sw$ in the direction w for every $s \in [0,1]$, with D continuous on $[0,1]$. If $W \subseteq F$ is closed, convex and contains $D(s)$ for every $s \in [0,1]$, then $f(z+w) - f(z) \in W$.*

Proof. Suppose not. By Hahn–Banach there are a continuous linear functional φ and $c \in \mathbb{R}$ with $\varphi(f(z+w) - f(z)) > c \ge \varphi(v)$ for all $v \in W$. The real function $g(s) = \varphi(f(z+sw))$ has derivative $\varphi(D(s)) \le c$ on $[0,1]$, and this derivative is continuous. By the fundamental theorem of calculus for real functions, $g(1) - g(0) = \int_0^1 \varphi(D(s))\,ds \le c$, a contradiction. □

This is the remark after Lemma 8.4 in [Mic80]. A first consequence is that the continuity required in Definition 2.3 is, from the first order on, automatic.

Lemma 2.9 (Continuity)**.** *If all directional derivatives of f exist on U and are jointly continuous on $U \times E$, then f is continuous on U.*

Proof. Since $\mathrm{d}f(x)(0) = 0$, joint continuity gives, for each closed convex neighborhood W of 0, a neighborhood of $(x, 0)$ on which $\mathrm{d}f$ takes values in W. For w small, the mean value inclusion then gives $f(x+w) - f(x) \in W$. This is [Mic80, 8.1]. □

When the derivatives are tested against continuous linear functionals, the mean value inclusion becomes an identity.

Proposition 2.10 (Mean value theorem, weak form)**.** *Let f be C^1 on a set containing the segment $\{x+th : t \in [0,1]\}$. Then, for every continuous linear functional φ on F,*

$$\varphi(f(x+h)) - \varphi(f(x)) = \int_0^1 \varphi(\mathrm{d}f(x+th)(h))\,dt.$$

Proof. The real function $t \mapsto \varphi(f(x+th))$ has continuous derivative $t \mapsto \varphi(\mathrm{d}f(x+th)(h))$; it suffices to apply the fundamental theorem of calculus for real functions. □

In the language of Chapter 4, $f(x+h) - f(x)$ is the weak integral of $t \mapsto \mathrm{d}f(x+th)(h)$ over $[0,1]$ (Corollary 4.9). This is [GN26, 1.2.6]. Note that no completeness of F is needed here: the integral exists because its value, $f(x+h) - f(x)$, is given in advance.

2.3 Glöckner's lemma and the linearity of the differential

The following lemma is the technical core of the chapter. It says that, for a C^1 map, the difference quotient is jointly continuous in the base point, the direction and the step, up to and including $t = 0$.

Lemma 2.11 (Glöckner)**.** *Let g be C^1 on U and $y \in U$, $v \in E$. Then*

$$\frac{1}{t}(g(y'+tv') - g(y')) \longrightarrow \mathrm{d}g(y)(v) \qquad \text{as } (y', v', t) \to (y, v, 0),\ t \neq 0.$$

Proof. By the mean value inclusion, the difference quotient lies in the closed convex hull of $\{\mathrm{d}g(y'+stv')(v') : s \in [0,1]\}$. Let W be a closed convex neighborhood of $\mathrm{d}g(y)(v)$. By continuity of $\mathrm{d}g$ at (y, v), all these derivatives lie in W when (y', v', t) is close to $(y, v, 0)$; hence so does the difference quotient. Since closed convex neighborhoods form a basis, the claim follows. □

In other words, the difference quotient extends continuously to $t = 0$ by $\mathrm{d}g$. This is the characterization of C^1 maps of Glöckner [Glo02] ([GN26, 1.2.10]), and it is the key step in the chain rule. It also gives the linearity of the differential.

Theorem 2.12 (The differential is linear)**.** *If f is C^1 on U and $x \in U$, then $h \mapsto \mathrm{d}f(x)(h)$ is a continuous linear map $E \to F$.*

Proof. Homogeneity, $\mathrm{d}f(x)(sh) = s\,\mathrm{d}f(x)(h)$, follows from the definition by a change of variable in the limit. For additivity we write

$$\tfrac{1}{t}(f(x+th+tk) - f(x)) = \tfrac{1}{t}(f((x+th)+tk) - f(x+th)) + \tfrac{1}{t}(f(x+th) - f(x)).$$

As $t \to 0$ the second term tends to $\mathrm{d}f(x)(h)$, and by Glöckner's lemma, applied at the moving base point $x+th \to x$, the first one tends to $\mathrm{d}f(x)(k)$. Continuity follows from the joint continuity of $\mathrm{d}f$. This is [GN26, 1.2.11]. □

Example 2.13. The existence of all directional derivatives, even everywhere and for a continuous map, does not imply linearity. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $f(a,b) = a^2b/(a^2+b^2)$ for $(a,b) \neq 0$ and $f(0) = 0$. It is continuous, since $|f(a,b)| \leq |b|$, and smooth away from the origin. Since $f(ta, tb) = tf(a,b)$, we have $\mathrm{d}f(0)(a,b) = f(a,b)$, which is not additive: $\mathrm{d}f(0)(1,0) = \mathrm{d}f(0)(0,1) = 0$, while $\mathrm{d}f(0)(1,1) = 1/2$. By Theorem 2.12, f is not C^1; since f is smooth away from the origin, what fails is the joint continuity of $\mathrm{d}f$ at some point of the form $(0,h)$.

2.4 The chain rule

Theorem 2.14 (Chain rule)**.** *Let $U \subseteq E$ and $V \subseteq F$ be open, $f\colon U \to F$ with $f(U) \subseteq V$ and $g\colon V \to G$. If f and g are C^k, then $g \circ f$ is C^k, and for $k \geq 1$*

$$\mathrm{d}(g \circ f)(x)(h) = \mathrm{d}g(f(x))(\mathrm{d}f(x)(h)).$$

Proof. For $t \neq 0$ write $f(x+th) = f(x) + t\,q(t)$, where $q(t) = \frac{1}{t}(f(x+th) - f(x)) \to \mathrm{d}f(x)(h)$. Then

$$\tfrac{1}{t}(g(f(x+th)) - g(f(x))) = \tfrac{1}{t}(g(f(x) + t\,q(t)) - g(f(x))),$$

which tends to $\mathrm{d}g(f(x))(\mathrm{d}f(x)(h))$ by Glöckner's lemma at $(y', v', t) = (f(x), q(t), t)$. This gives the formula. It says that $\mathrm{d}(g\circ f) = \mathrm{d}g \circ (f \circ \pi_1, \mathrm{d}f)$, a composition of C^{k-1} maps, so the regularity follows by induction on k. This is [GN26, 1.2.13 and 1.3.4]. □

Remark 2.15. The direction $q(t)$ at which Glöckner's lemma is applied moves with t. This is where the continuity of $\mathrm{d}g$ in the direction, and not only in the base point, enters the proof.

2.5 Partial derivatives

As in finite dimension, a map on a product is C^1 as soon as its partial derivatives exist and are continuous.

Theorem 2.16 (Continuous partial derivatives)**.** *Let $U \subseteq E \times F$ be open and $f\colon U \to G$. Suppose that the partial derivatives $\mathrm{d}_1 f(p)(u) = \mathrm{d}f(p)(u, 0)$ and $\mathrm{d}_2 f(p)(v) = \mathrm{d}f(p)(0, v)$ exist for $p \in U$, and are jointly continuous on $U \times E$ and $U \times F$. Then f is C^1 on U, and $\mathrm{d}f(p)(u, v) = \mathrm{d}_1 f(p)(u) + \mathrm{d}_2 f(p)(v)$.*

Proof. We split

$$f(p + t(u, v)) - f(p) = [f(p + t(u, v)) - f(p + t(0, v))] + [f(p + t(0, v)) - f(p)].$$

Divided by t, the second bracket tends to $\mathrm{d}_2 f(p)(v)$. By the mean value inclusion, the first one, divided by t, lies in the closed convex hull of $\{\mathrm{d}_1 f(p + t(0, v) + st(u, 0))(u) : s \in [0, 1]\}$, and these values tend to $\mathrm{d}_1 f(p)(u)$ by joint continuity. The resulting formula for $\mathrm{d}f$ is jointly continuous, and f is continuous by Lemma 2.9. This is [Mic80, 8.3] and [GN26, 1.2.8]. □

This criterion is a practical way to verify that a given map is C^1; for instance, it is used in the proof that continuous bilinear maps are smooth (Theorem 3.9).

2.6 Symmetry of the second differential

For a C^2 map, the second differential at x is $(h_1, h_2) \mapsto \mathrm{d}(\mathrm{d}f)((x, h_1), (h_2, 0))$, the derivative of $\mathrm{d}f(\cdot)(h_1)$ in the direction h_2. We prove that it is symmetric. As in the classical proof, both orders of differentiation are compared with a second difference quotient, which is symmetric by construction.

Lemma 2.17 (Second differences)**.** *Let f be C^2 on U, $x \in U$ and $h_1, h_2 \in E$. For every closed convex neighborhood W of $\mathrm{d}(\mathrm{d}f)((x, h_1), (h_2, 0))$, the second difference quotient*

$$\frac{1}{t^2}(f(x + th_1 + th_2) - f(x + th_1) - f(x + th_2) + f(x))$$

lies in W for all small $t \neq 0$.

Proof. We apply the mean value inclusion twice: first to $s \mapsto f(x + sh_1 + th_2) - f(x + sh_1)$, then to the derivative of this map in the direction h_2. The second difference quotient lies in the closed convex hull of values of the second differential at points near x, and these lie in W by continuity. □

Theorem 2.18 (Schwarz, second order)**.** *If f is C^2 on U and $x \in U$, then $\mathrm{d}(\mathrm{d}f)((x, h_1), (h_2, 0)) = \mathrm{d}(\mathrm{d}f)((x, h_2), (h_1, 0))$ for all $h_1, h_2 \in E$.*

Proof. The second difference quotient is symmetric in h_1 and h_2. By the previous lemma it converges to both sides, and limits in a Hausdorff space are unique. This is [GN26, 1.3.17] for $k = 2$. □

The symmetry of the higher differentials is the subject of Chapter 3.

2.7 Fréchet differentiable maps

On normed spaces both calculi are available, and the Fréchet one is the stronger of the two.

Proposition 2.19 (Fréchet C^k implies Michal–Bastiani C^k)**.** *Let E and F be normed spaces. If f is C^k on U in the sense of Fréchet, then it is C^k in the sense of Michal–Bastiani.*

Proof. A Fréchet differentiable map has all directional derivatives, with $\mathrm{d}f(x)(h) = Df(x)h$. The map $(x, h) \mapsto Df(x)h$ is Fréchet C^{k-1} on $U \times E$, and the claim follows by induction on k. This is [Nee06, Remark I.2.2]. □

The converse fails for C^k, but holds up to the loss of one derivative; see Chapter 8.

Chapter 3

Higher differentials and rules of calculus

There are two natural ways to say that a map $f\colon U \to F$ is C^k. The one we adopted in Definition 2.3 is recursive: f is C^{k+1} when $\mathrm{d}f$, regarded as a map on the open set $U \times E$, is C^k. This is the form used by Glöckner, and it is the convenient one for proofs by induction. The classical definition, due to Bastiani [Bas64] in 1964, asks instead that the iterated directional derivatives $\mathrm{d}^{(j)}f(x; y_1, \dots, y_j)$ exist and be jointly continuous in the point and in all the directions, for $j \le k$.

In this chapter we prove that both definitions agree, and we derive from this the basic structure of the higher differentials: if f is C^{k+r}, then $\mathrm{d}^{(k)}f$ is again C^r, and for each x the map $\mathrm{d}^{(k)}f(x;\cdot)$ is continuous, symmetric and k-linear. We first define the iterated differentials and Bastiani's notion, and prove a formula (the slice lemma) relating the iterated differentials of f to those of $\mathrm{d}f$; this formula is the bridge between the two definitions. We then deal with regularity and symmetry, and the last section collects the rules of calculus used in practice: the product rule, Taylor's formula and the second-order chain rule. Our reference is [GN26, §1.3].

3.1 Iterated differentials

Definition 3.1 (Iterated differentials)**.** For $f\colon E \to F$, $x \in E$ and $y_1, \dots, y_k \in E$ we define

$$\mathrm{d}^{(0)}f(x) = f(x), \qquad \mathrm{d}^{(k+1)}f(x; y_1, \dots, y_{k+1}) = \mathrm{d}\big(z \mapsto \mathrm{d}^{(k)}f(z; y_1, \dots, y_k)\big)(x)(y_{k+1}),$$

that is, $\mathrm{d}^{(k)}f(x; y_1, \dots, y_k) = (D_{y_k} \cdots D_{y_1} f)(x)$.

So $\mathrm{d}^{(k)}f(x; y_1, \dots, y_k)$ is obtained by differentiating f successively in the directions $y_1, \dots, y_k$, keeping the directions fixed and moving only the point. Note that $\mathrm{d}^{(1)}f(x; y) = \mathrm{d}f(x)(y)$.

Definition 3.2 (Bastiani's C^k maps)**.** A map f has continuous iterated derivatives of order k on U if, for every $j \le k$, the iterated differentials $\mathrm{d}^{(j)}f(x; y_1, \dots, y_j)$ exist for $x \in U$ and are continuous on $U \times E^j$.

Remark 3.3. In a Banach space the k-th Fréchet derivative is a map from U into the space of continuous k-linear maps $E^k \to F$, and its continuity is measured in the operator norm. Here no topology is put on spaces of multilinear maps: what is required is the joint continuity of $\mathrm{d}^{(j)}f$ on $U \times E^j$. This is the same choice already made for $k = 1$ in Definition 2.3, and it is what allows the theory to work without norms.

The bridge between the two definitions is a formula that expresses the iterated differentials of f through those of $\mathrm{d}f$, in directions of the form $(y, 0)$.

Lemma 3.4 (Slices)**.** *For $g\colon E \times E' \to F$ and $a \in E'$,*

$$\mathrm{d}^{(k)}(g(\cdot, a))(x; v_1, \dots, v_k) = \mathrm{d}^{(k)}g((x, a); (v_1, 0), \dots, (v_k, 0)).$$

In particular

$$\mathrm{d}^{(k+1)}f(x; y_0, \dots, y_k) = \mathrm{d}^{(k)}(\mathrm{d}f)((x, y_0); (y_1, 0), \dots, (y_k, 0)).$$

In words: differentiating the partial map $g(\cdot, a)$ is the same as differentiating g in directions that do not move the second variable. The second formula is the case $g = \mathrm{d}f$, $a = y_0$.

Proof. Both sides are limits of the same difference quotients, and the claim follows by induction on k. The second formula is [GN26, (1.36)]. □

Theorem 3.5 (The two definitions of C^k agree)**.** *A map f is C^k on U in the sense of Definition 2.3 if and only if it has continuous iterated derivatives of order k on U.*

From now on we use whichever of the two descriptions is more convenient.

Proof. Both implications go by induction on k, through Lemma 3.4. If f is C^{k+1}, the iterated differentials of f are slices of those of $\mathrm{d}f$, which is C^k; hence they exist and are continuous. For the converse, the derivatives of $\mathrm{d}f$ in a general direction (h, w) split, by linearity in the second variable and Theorem 2.16, into a derivative in the direction $(h, 0)$, which is an iterated differential of f, and one in the direction $(0, w)$, which is again of the form $\mathrm{d}f$. The details can be found in [GN26, 1.3.10 and 1.3.12]. □

3.2 Regularity and symmetry

Theorem 3.6 (Regularity of the differentials)**.** *If f is C^{k+r} on U, then $\mathrm{d}^{(k)}f$ is C^r on $U \times E^k$.*

Thus each differentiation costs exactly one order of regularity, as in the Banach case; in particular, the iterated differentials of a smooth map are smooth.

Proof. We argue by induction on k, using that $\mathrm{d}^{(k+1)}f$ is, up to reordering the variables, the derivative of $\mathrm{d}^{(k)}f$ in directions of the form $(y, 0)$. This is [GN26, 1.3.13]. □

Theorem 3.7 (Schwarz)**.** *If f is C^k on U and $x \in U$, then $\mathrm{d}^{(k)}f(x; \cdot)$ is a continuous symmetric k-linear map $E^k \to F$.*

This is what allows us to regard $\mathrm{d}^{(k)}f(x)$ as the k-th differential of f at x, in the same sense as in finite dimension: the order in which we differentiate is irrelevant, and the result depends linearly on each direction.

Proof. We begin with symmetry. It suffices to consider transpositions of adjacent variables, and these reduce to Theorem 2.18 applied to an iterated differential of lower order. Linearity in the last variable is Theorem 2.12 applied to $\mathrm{d}^{(k-1)}f$; by symmetry, the map is linear in every variable. Continuity follows from the joint continuity of $\mathrm{d}^{(k)}f$. This is [GN26, 1.3.17]. □

Remark 3.8. A word of caution: the continuity in Theorem 3.7 is continuity of the k-linear map $\mathrm{d}^{(k)}f(x; \cdot)$ on E^k, for a fixed x. Nothing is claimed about the dependence of this multilinear map on x in a topology on a space of multilinear maps; the regularity in x is expressed by the joint statement of Theorem 3.6.

3.3 Rules of calculus

With the chain rule (Theorem 2.14) and the structure of the higher differentials (Chapter 3) at hand, we can now derive the remaining rules of calculus that are used in practice. All of them are classical in the Banach setting, and their statements are the familiar ones; what has to be checked is that the proofs survive without norms. In each case the argument either reduces to the chain rule or, through a continuous linear functional, to the calculus of real functions of one variable.

We begin with continuous bilinear maps and the product rule. We then compute the derivatives of a map along a line, which leads to Taylor's formula in a weak form, valid without any completeness assumption on F. The section closes with the chain rule for second differentials. The references are the results of [GN26] quoted in each proof.

3.3.1 Bilinear maps and the product rule

Theorem 3.9 (Bilinear maps)**.** *Let $\beta\colon A \times B \to F$ be continuous and bilinear. Then β is C^k for every k, with $\mathrm{d}\beta((a,b),(h,w)) = \beta(a,w) + \beta(h,b)$.*

Proof. The difference quotient of β at (a,b) in the direction (h,w) is $\beta(a,w)+\beta(h,b)+t\,\beta(h,w)$, and letting $t \to 0$ gives the formula. The differential is therefore a sum of continuous bilinear maps composed with continuous linear maps, and induction on k completes the proof. This is [GN26, 1.2.3 and 1.3.5]. □

Note that continuity of β means joint continuity on $A \times B$. This cannot be weakened to separate continuity: for $b = 0$ and $h = 0$ the differential above reduces to $\beta(a,w)$, which must be jointly continuous in (a,w).

Corollary 3.10 (Product rule)**.** *If $f\colon U \to A$ and $g\colon U \to B$ are C^k and β is continuous and bilinear, then $x \mapsto \beta(f(x),g(x))$ is C^k and*

$$\mathrm{d}(\beta(f,g))(x)(h) = \beta(f(x),\mathrm{d}g(x)(h)) + \beta(\mathrm{d}f(x)(h),g(x)).$$

Proof. It is immediate from the chain rule applied to $\beta \circ (f,g)$, together with Theorem 3.9. □

Example 3.11. Let $\beta\colon E \times E \to F$ be continuous and bilinear, and let $q(x) = \beta(x,x)$. Taking $f = g = \mathrm{id}_E$ in Corollary 3.10, we obtain that q is smooth, with $\mathrm{d}q(x)(h) = \beta(x,h) + \beta(h,x)$. When β is symmetric this is the familiar $2\beta(x,h)$.

3.3.2 Taylor's formula

To pass from the differentials of f to the values of f we restrict f to a line and compose with a continuous linear functional. The resulting real function of one variable is as regular as f, and its derivatives are the iterated differentials of f along the line.

Lemma 3.12 (Derivatives along a line)**.** *Let f be C^k on U, $j < k$ and ψ a continuous linear functional on F. Then the real function $s \mapsto \psi(\mathrm{d}^{(j)}f(x+sh;h,\dots,h))$ has derivative $\psi(\mathrm{d}^{(j+1)}f(x+sh;h,\dots,h))$ wherever $x + sh \in U$.*

Proof. By the chain rule along the line $s \mapsto (x+sh,h,\dots,h)$, whose derivative is $(h,0,\dots,0)$. □

Theorem 3.13 (Taylor's formula, weak form)**.** *Let f be C^{k+1} on a set containing the segment $[x, x+h]$. For every continuous linear functional ψ on F,*

$$\psi(f(x+h)) = \sum_{j=0}^{k} \frac{1}{j!}\,\psi(\mathrm{d}^{(j)} f(x; h, \dots, h)) + \int_0^1 \frac{(1-t)^k}{k!}\,\psi(\mathrm{d}^{(k+1)} f(x+th; h, \dots, h))\,dt.$$

Proof. We apply Taylor's formula with integral remainder to the real function $s \mapsto \psi(f(x+sh))$, whose derivatives are given by Lemma 3.12. Compare [GN26, 1.6.1 and 1.6.14], where the formula is stated for the integrals in F; the weak form needs no completeness of F. □

Remark 3.14. The remainder in Theorem 3.13 is a real integral, one for each functional ψ. Writing it as a vector of F requires the weak integral of $t \mapsto \frac{(1-t)^k}{k!}\,\mathrm{d}^{(k+1)} f(x+th; h, \dots, h)$ to exist, which in general depends on a completeness property of F (Chapters 4 and 5). Since continuous linear functionals separate points, the weak form determines $f(x+h)$ all the same.

3.3.3 The second-order chain rule

Theorem 3.15 (Second-order chain rule)**.** *Let $\tau\colon U \to V$ and $g\colon V \to F$ be C^2, with $U \subseteq E$ and $V \subseteq E'$ open. For $y \in U$ and $a, b \in E$,*

$$\mathrm{d}^{(2)}(g \circ \tau)(y; a, b) = \mathrm{d}^{(2)} g(\tau y; \mathrm{d}\tau(y)(a), \mathrm{d}\tau(y)(b)) + \mathrm{d}g(\tau y)(\mathrm{d}^{(2)}\tau(y; a, b)).$$

The first term is what one would guess by applying the chain rule twice naively; the second involves the second differential of τ, and vanishes when τ is the restriction of a continuous linear map, since then $\mathrm{d}\tau(y)(b)$ does not depend on y.

Proof. On $U \times E$, the first-order chain rule gives $\mathrm{d}(g\circ\tau) = \mathrm{d}g \circ \Phi$ with $\Phi(y', b') = (\tau y', \mathrm{d}\tau(y')(b'))$. We differentiate this once more with the first-order chain rule. The derivative of Φ at (y, b) in the direction $(a, 0)$ is $(\mathrm{d}\tau(y)(a), \mathrm{d}^{(2)}\tau(y; a, b))$. Splitting it as $(\mathrm{d}\tau(y)(a), 0) + (0, \mathrm{d}^{(2)}\tau(y; a, b))$, the first term gives $\mathrm{d}^{(2)} g$, and the second gives $\mathrm{d}g$, because $\mathrm{d}g(\tau y)$ is linear. □

3.3.4 Operations on C^k maps

Proposition 3.16 (Operations)**.** *Directional derivatives and C^k maps behave as expected under the elementary operations: maps into products are differentiated componentwise, and finite sums, multiples, differences, products $f \cdot g$ of a real function with a vector-valued one, and products of maps $(x, y) \mapsto (g(x), h(y))$ of C^k maps are C^k. A C^k map with values in a closed subspace is C^k into the subspace [GN26, compare Lemma 1.3.19]. The differential depends only on the germ of the map, and on normed spaces the first and second Michal–Bastiani differentials of C^1 and C^2 maps are the Fréchet derivatives.*

Proof. Each statement reduces to the chain rule and continuous linear maps, or to the product rule for the continuous bilinear map $(t, v) \mapsto tv$; the closed subspace case uses that difference quotients stay in the subspace. □

Chapter 4

Weak integrals

Integration enters Michal–Bastiani calculus in two places: through the mean value theorem, which expresses $f(x+h)-f(x)$ as an integral of the differential along the segment, and through integrals depending on parameters, which are a basic tool to produce smooth maps. Without a norm there is no Bochner integral, and without completeness Riemann sums need not converge. The integral used in infinite-dimensional Lie theory is therefore defined by duality: a vector is the integral of a curve if every continuous linear functional sees it as such.

Weak integrals defined in this way are unique, by Hahn–Banach, but they need not exist; existence is the subject of the next chapter (Chapter 5). Everything in the present chapter holds in an arbitrary locally convex space, with the existence of the relevant integrals as an explicit hypothesis. We first establish the basic properties of the integral and the change of variables formula, then the first part of the fundamental theorem of calculus, which needs no hypothesis on F at all, and finally the continuity and differentiability of integrals depending on parameters. Our reference is [GN26, §1.1].

4.1 Definition and basic properties

Definition 4.1 (Weak integral)**.** Let $\gamma\colon \mathbb{R} \to F$ and $a, b \in \mathbb{R}$. A vector $y \in F$ is a weak integral of γ over $[a,b]$ if

$$\varphi(y) = \int_a^b \varphi(\gamma(t))\,dt$$

for every continuous linear functional φ on F.

Lemma 4.2 (Uniqueness)**.** *Weak integrals are unique.*

Proof. Two weak integrals have the same image under every continuous linear functional, and these separate points by Hahn–Banach. □

We write $\int_a^b \gamma$ for the weak integral when it exists.

Remark 4.3. In Lean, the function that returns the weak integral is total: its value is 0 when the integral does not exist. It is only used where existence is known.

Lemma 4.4 (Algebraic rules)**.** *For continuous curves, weak integrals are linear in the integrand and additive in the interval: $\int_a^b \gamma + \int_b^c \gamma = \int_a^c \gamma$. The constant curve v has integral $(b-a)v$. If $L\colon F \to G$ is continuous and linear, then $L(\int_a^b \gamma)$ is the weak integral of $L \circ \gamma$.*

Proof. Apply a functional and use the corresponding rules for real integrals. For L, it suffices to note that $\varphi \circ L$ is a continuous linear functional on F whenever φ is one on G. This is [GN26, 1.1.4 and 1.1.8]. □

The next lemma replaces the estimate $\| \int_a^b \gamma \| \le (b-a) \sup \|\gamma\|$ of the normed case. Since there is no norm, the size of a vector is measured by the closed convex neighborhoods of 0 that contain it.

Lemma 4.5 (Integrals of small curves are small)**.** *Let $a \le b$ and let γ be continuous on $[a,b]$ with weak integral y. If $W \subseteq F$ is closed, convex, contains 0 and contains $\gamma([a,b])$, then $y \in (b-a+1)\,W$.*

Proof. If not, Hahn–Banach separates y from the closed convex set $(b-a+1)W$ by a functional φ. Bounding $\int_a^b \varphi(\gamma)$ by $(b-a)\sup_W \varphi$ gives a contradiction. The constant $b-a+1$ instead of $b-a$ avoids a separate treatment of the case $a=b$. Compare [GN26, 1.1.9]. □

Remark 4.6. When F is normed and W is the closed ball of radius r centered at 0, the lemma says that $\|y\| \le (b-a+1)\,r$ whenever $\|\gamma(t)\| \le r$ on $[a,b]$. Up to the harmless constant, this is the classical estimate. For the applications in this document (continuity and convergence statements) only the shape of the bound matters, not the constant.

4.2 Change of variables

Proposition 4.7 (Change of variables)**.** *Let ϕ be a real C^1 function on $[a,b]$ and γ continuous on $\phi([a,b])$. Then y is a weak integral of $t \mapsto \phi'(t)\,\gamma(\phi(t))$ over $[a,b]$ if and only if it is a weak integral of γ over $[\phi(a),\phi(b)]$.*

Note that the statement is an equivalence: the existence of either integral implies the existence of the other. The case used most often is the affine one, $\phi(t) = a + t(b-a)$ on $[0,1]$, which shows that $\int_a^b \gamma$ exists if and only if $\int_0^1 (b-a)\,\gamma(a+t(b-a))\,dt$ does, and that both coincide.

Proof. Apply a functional and use the change of variables formula for real integrals. □

4.3 The fundamental theorem of calculus, first part

Theorem 4.8 (Fundamental theorem of calculus, I)**.** *Let γ be differentiable on $[a,b]$, with continuous derivative γ'. Then $\gamma(b)-\gamma(a)$ is the weak integral of γ' over $[a,b]$.*

Proof. For every functional φ, the real function $\varphi \circ \gamma$ has continuous derivative $\varphi \circ \gamma'$, and it suffices to apply the real fundamental theorem of calculus. This is [GN26, 1.1.5]. □

We remark that no completeness of F is needed: the integral exists because $\gamma(b)-\gamma(a)$ is a candidate. In other words, derivatives of C^1 curves are always integrable; the difficulty of the existence problem lies with curves that are not known to be derivatives.

Corollary 4.9 (Mean value theorem)**.** *If f is C^1 on a set containing the segment $[x, x+h]$, then $f(x+h) - f(x) = \int_0^1 \mathrm{d}f(x+th)(h)\,dt$, the weak integral existing in F.*

Proof. This is Proposition 2.10, read as a statement about weak integrals. See [GN26, 1.2.6]. □

4.4 Integrals depending on parameters

We now let the integrand depend on a parameter x and study the regularity of $x \mapsto \int_a^b f(x,t)\,dt$. As in the classical case, the arguments rest on the compactness of $[a,b]$, which turns pointwise closeness into uniform closeness in t; the uniform estimate is then transferred to the integrals by Lemma 4.5.

Proposition 4.10 (Continuity)**.** *Let X be a topological space, $U \subseteq X$ and $T \subseteq \mathbb{R}$ open with $[a,b] \subseteq T$, and let $f\colon U \times T \to F$ be continuous. If the weak integrals $I(x) = \int_a^b f(x,t)\,dt$ exist for $x \in U$, then I is continuous on U.*

Proof. By compactness of $[a,b]$ (the tube lemma), for x' close to x the curve $f(x',\cdot) - f(x,\cdot)$ takes values in a given closed convex neighborhood W of 0 on $[a,b]$. By Lemma 4.5, $I(x') - I(x) \in (b-a+1)W$. Since the closed convex neighborhoods of 0 form a basis, this proves the claim. This is [GN26, 1.1.11]. □

Theorem 4.11 (Differentiation under the integral sign)**.** *Let $f\colon U \times T \to F$ be continuous, with partial derivatives $D(x,t)(h) = \mathrm{d}f((x,t),(h,0))$ jointly continuous on $(U \times T) \times E$. Suppose that the weak integrals $I(x) = \int_a^b f(x,t)\,dt$ exist for $x \in U$, and that $J = \int_a^b D(x,t)(h)\,dt$ exists. Then I has derivative J at x in the direction h.*

In other words, $\mathrm{d}I(x)(h) = \int_a^b D(x,t)(h)\,dt$, as in the classical case, provided that both sides make sense. The existence of the integrals is part of the hypotheses; in a space with a suitable completeness property it is automatic (Chapter 5).

Proof. For $s \neq 0$, the difference quotient $\frac{1}{s}(I(x+sh) - I(x))$ is the weak integral of $t \mapsto \frac{1}{s}(f(x+sh,t) - f(x,t))$. By the mean value inclusion and the tube lemma, for small s this curve is uniformly close to $t \mapsto D(x,t)(h)$ on $[a,b]$, and Lemma 4.5 gives the convergence. This is [GN26, 1.3.15 and 1.3.23]. □

Chapter 5

Completeness, Mackey completeness and the existence of integrals

In Chapter 4 the weak integral of a curve was defined, but its existence was always a hypothesis. In a Banach space this hypothesis is superfluous: continuous curves on a compact interval have Riemann integrals. In a general locally convex space nothing of the sort holds, and one has to pay for integrals with some completeness of F. The aim of this chapter is to determine how much completeness is needed, and to draw the consequences for integrals depending on parameters.

Two answers are classical. Weak integrals of continuous curves exist when F is sequentially complete, and weak integrals of C^1 curves exist under the weaker assumption of Mackey completeness. The second one is the natural hypothesis for smooth calculus: it is what makes parameter-dependent integrals of smooth maps smooth. Mackey completeness is also the completeness notion of Kriegl and Michor [KM97, §2]; we stress that here it is only a hypothesis on the model space, inside Michal–Bastiani calculus. Our main reference is [GN26, §1.1].

The chapter is organized as follows. In Section 5.1 we introduce both completeness notions and compare them. In Section 5.2 we study Riemann sums, which in both cases provide the integral as a limit. Section 5.3 contains the two existence theorems, Section 5.4 the second half of the fundamental theorem of calculus, Section 5.5 the regularity of integrals depending on parameters, and Section 5.6 the equivalence of two definitions of Mackey completeness.

5.1 Completeness notions

We state completeness through the topology of F, without fixing a uniform structure.

Definition 5.1 (Sequential completeness)**.** A sequence (u_n) in F is Cauchy if for every neighborhood W of 0 there is N with $u_n - u_m \in W$ for all $m, n \geq N$. The space F is sequentially complete if every Cauchy sequence converges.

Definition 5.2 (Mackey completeness)**.** A sequence (u_n) in F is Mackey–Cauchy if there are a bounded set $B \subseteq F$ and real numbers $r_k \to 0$ with $u_l - u_k \in r_k B$ for all $k \leq l$. The space F is Mackey complete if every Mackey–Cauchy sequence converges.

Here bounded means bounded in the sense of von Neumann: absorbed by every neighborhood of 0. In other words, a Mackey–Cauchy sequence is Cauchy at a controlled rate, measured by a

single bounded set; in a normed space one may take for B the unit ball, and the condition reads $\|u_l - u_k\| \le r_k$.

Remark 5.3. Definition 5.2 is one of the equivalent descriptions of Mackey completeness in [GN26, 1.1.18(e)] and [KM97, 2.2, 2.14]. Glöckner and Neeb take as definition the existence of weak integrals of Lipschitz curves; we prove in Section 5.6 that both definitions agree, so the choice is a matter of convenience.

Proposition 5.4 (Comparison)**.** *Complete uniform additive groups are sequentially complete, and sequentially complete spaces are Mackey complete.*

In particular, every complete locally convex space (for instance, a Fréchet or a Banach space) is Mackey complete, and all the results of this chapter apply to it.

Proof. The first claim amounts to comparing our notion of Cauchy sequence with the uniform one. For the second, it suffices to observe that a Mackey–Cauchy sequence is Cauchy: given a neighborhood W of 0, the bounded set B is absorbed by W, so $r_k B \subseteq W$ for large k. □

We will need the following elementary fact, which is where local convexity enters.

Lemma 5.5 (Convex hulls of bounded sets)**.** *In a locally convex space, the convex hull of a bounded set is bounded.*

Proof. It is immediate that if B is absorbed by a convex balanced neighborhood W of 0, so is $\operatorname{conv} B$; and these neighborhoods form a basis of neighborhoods of 0. □

5.2 Riemann sums

In both existence theorems the integral will be obtained as the limit of a sequence of Riemann sums. We only need sums over uniform partitions.

Definition 5.6 (Riemann sums)**.** For $\gamma\colon \mathbb{R} \to F$ and $N \ge 1$, the Riemann sum of γ over $[a,b]$ with N equal subintervals and left endpoints is

$$R_N(\gamma) = \frac{b-a}{N} \sum_{i=0}^{N-1} \gamma\Big(a + i\,\frac{b-a}{N}\Big).$$

The basic estimate compares a Riemann sum with the sums of its refinements. It is stated with an arbitrary convex set V, so that the same lemma serves both for neighborhoods of 0 (the sequentially complete case) and for multiples of a bounded set (the Mackey complete case).

Lemma 5.7 (Refinements)**.** *Let $V \subseteq F$ be convex, and suppose that $\gamma(s) - \gamma(t) \in V$ whenever $s, t \in [a,b]$ and $0 \le s - t \le \frac{b-a}{N}$. Then $R_{Nm}(\gamma) - R_N(\gamma) \in (b-a)\,V$ for every $m \ge 1$. Moreover, for a continuous real function g the Riemann sums $R_N(g)$ tend to $\int_a^b g$.*

Proof. The difference $R_{Nm} - R_N$ is $b-a$ times an average of increments of γ over intervals of length at most $\frac{b-a}{N}$; since V is convex, the average lies in V. The second claim is the classical convergence of Riemann sums of continuous functions. □

Lemma 5.8 (Limits of Riemann sums)**.** *If γ is continuous on $[a,b]$ and the dyadic Riemann sums $R_{2^n}(\gamma)$ converge to y, then y is the weak integral of γ over $[a,b]$.*

Proof. For each continuous linear functional φ we have $\varphi(R_{2^n}(\gamma)) = R_{2^n}(\varphi \circ \gamma)$, which converges both to $\varphi(y)$ and, by Lemma 5.7, to $\int_a^b \varphi \circ \gamma$. □

Remark 5.9. The dyadic sums $R_{2^n}(\gamma)$ form a *sequence*, each term being a refinement of the previous one. This is why sequential notions of completeness suffice: no nets of partitions are needed.

5.3 Existence of integrals

We can now state the two existence theorems. The construction is the same in both; only the Cauchy estimate for the dyadic sums changes.

Theorem 5.10 (Continuous curves, sequentially complete case)**.** *If F is sequentially complete, every curve that is continuous on $[a, b]$ has a weak integral over $[a, b]$.*

Proof. By uniform continuity on the compact interval $[a, b]$, for each closed convex neighborhood V of 0 the increments of γ over sufficiently short intervals lie in V. By Lemma 5.7, the dyadic Riemann sums form a Cauchy sequence; its limit is the weak integral by Lemma 5.8. This is [GN26, 1.1.15]. □

Theorem 5.11 (C^1 curves, Mackey complete case)**.** *If F is Mackey complete, every curve that is C^1 on a neighborhood of $[a, b]$ has a weak integral over $[a, b]$.*

Proof. Let K be the closure of the convex hull of $\{s\,\gamma'(t) : s \in [0, 1],\ t \in [a, b]\}$. It is bounded by Lemma 5.5, since $\gamma'([a, b])$ is compact. By the mean value inclusion (Lemma 2.8), $\gamma(s) - \gamma(t) \in h\,K$ whenever $0 \le s - t \le h$. Then Lemma 5.7 gives

$$R_{2^m}(\gamma) - R_{2^n}(\gamma) \in \frac{(b-a)^2}{2^n}\,K \qquad (n \le m),$$

so the dyadic Riemann sums form a Mackey–Cauchy sequence, with bounded set K and $r_n = (b-a)^2 2^{-n}$. Its limit is the weak integral by Lemma 5.8. Compare [GN26, 1.1.18] and [KM97, 2.14]. □

Remark 5.12. In the continuous case the Cauchy estimate comes from uniform continuity, and it only says that the differences of dyadic sums are eventually small. In the C^1 case it comes from the mean value inclusion, and it says more: the differences lie in multiples of one fixed bounded set, with factors tending to 0 at a known rate. This additional control is what allows us to replace sequential completeness by Mackey completeness.

5.4 The fundamental theorem of calculus, second part

The first part of the fundamental theorem (Theorem 4.8) recovers a C^1 curve from the integral of its derivative. The second part, which we prove now, differentiates an integral with respect to its upper limit.

Theorem 5.13 (Fundamental theorem of calculus, II)**.** *Let $I \subseteq \mathbb{R}$ be an open interval, $a \in I$ and γ continuous on I. If the weak integrals $\eta(s) = \int_a^s \gamma$ exist for $s \in I$, then η is differentiable on I with $\eta' = \gamma$. If F is sequentially complete, these integrals always exist.*

Proof. By additivity in the interval and the affine change of variables $r \mapsto t + sr$,

$$\frac{1}{s}(\eta(t+s) - \eta(t)) - \gamma(t) = \int_0^1 (\gamma(t+sr) - \gamma(t))\, dr.$$

Let W be a closed convex neighborhood of 0. By continuity of γ at t, for small s the integrand lies in W, and Lemma 4.5 puts the integral in $2W$. The last claim follows from Theorem 5.10. This is [GN26, 1.1.12]. □

Remark 5.14. Note that the first assertion needs no completeness at all: what is assumed is the existence of the integrals $\eta(s)$, and the limit of the difference quotients is then controlled by Hahn–Banach through Lemma 4.5. Completeness of F enters only to guarantee that existence.

5.5 Integrals depending on parameters

Let U be an open subset of a locally convex space E, let $T \subseteq \mathbb{R}$ be open with $[a, b] \subseteq T$, and let f be a map on $U \times T$. We study the regularity of $x \mapsto \int_a^b f(x,t)\, dt$. All the results follow from a single induction, in which the existence of the integrals is isolated as a hypothesis.

Theorem 5.15 (The induction behind parameter-dependent integrals)**.** *Let $T \subseteq \mathbb{R}$ be open with $[a, b] \subseteq T$, and $m \in \mathbb{N}$. Suppose that, for every C^m map f on a product $U \times T$, the weak integrals $\int_a^b f(x,t)\, dt$ exist. Then, for every C^{k+m} map f on $U \times T$, the map $x \mapsto \int_a^b f(x,t)\, dt$ is C^k on U.*

Here the hypothesis is understood for all open sets U of all locally convex spaces E; this uniformity is what makes the induction work.

Proof. We argue by induction on k, for all locally convex spaces E at once. The case $k = 0$ is Proposition 4.10. For the inductive step, differentiation under the integral sign (Theorem 4.11) gives

$$\mathrm{d}\Big(\int_a^b f(\cdot,t)\, dt\Big)(x)(h) = \int_a^b \mathrm{d}f((x,t),(h,0))\, dt,$$

and the integrand is a C^{k-1+m} map on the product $(U \times E) \times T$, to which the inductive hypothesis applies. □

Remark 5.16. This single induction yields all the results on parameter-dependent integrals at once; the sources prove the sequentially complete case and the Mackey complete case separately.

Remark 5.17 (Lean). The quantification over all spaces E is the reason for the universe restriction mentioned in Chapter 2.

Combined with the existence theorems of Section 5.3, the induction gives the following statements, with no existence hypotheses left.

Corollary 5.18 (Parameter-dependent integrals)**.** *Let f be a map on $U \times T$ with $[a, b] \subseteq T$, and $I(x) = \int_a^b f(x,t)\, dt$.*

1. *If F is sequentially complete and f is C^k, then I is C^k, and $\mathrm{d}I(x)(h) = \int_a^b \mathrm{d}f((x,t),(h,0))\, dt$ for $k \geq 1$.*
2. *If F is Mackey complete and f is C^{k+1}, then I is C^k; if f is C^2, the formula for $\mathrm{d}I$ of (1) holds.*

3. If F is Mackey complete and f is smooth, then I is smooth.

Proof. Apply Theorem 5.15 with $m = 0$ and Theorem 5.10 in the first case, and with $m = 1$ and Theorem 5.11 in the other two. This is [GN26, 1.3.23]. □

Remark 5.19. The loss of one derivative in the Mackey complete case has a simple origin: in that setting we only know that C^1 curves have integrals (Theorem 5.11), so the induction has to be started with $m = 1$. For smooth maps nothing is lost, and this is why Mackey completeness is the natural hypothesis for smooth calculus.

5.6 Mackey completeness and Lipschitz curves

There are two current definitions of Mackey completeness. The one we adopted in Definition 5.2 asks for the convergence of Mackey–Cauchy sequences, as in [KM97, 2.2]. Glöckner and Neeb, on the other hand, define Mackey completeness through integrals: a space is Mackey complete if every Lipschitz curve has a weak integral ([GN26, 1.1.17(b)]). Their definition says directly which curves can be integrated; ours is phrased in terms of sequences alone. In this section we prove that both definitions agree.

The equivalence is stated in [GN26, 1.1.18(e)] and [KM97, 2.14]. One direction is a variant of the existence theorem of Chapter 5, with difference quotients in place of derivatives. For the converse we give our own argument: a Mackey–Cauchy sequence is encoded as a Lipschitz curve whose integral is the limit we are looking for.

Definition 5.20 (Lipschitz curves)**.** A curve $\gamma\colon [a,b] \to F$ is Lipschitz if the set of difference quotients

$$\left\{ \tfrac{1}{s-t}(\gamma(s) - \gamma(t)) : s, t \in [a,b],\ s \neq t \right\}$$

is bounded in F.

In a normed space this is the usual Lipschitz condition. In general there is no constant to speak of, and its role is played by the bounded set of difference quotients.

Definition 5.21 (Mackey completeness after Glöckner and Neeb)**.** We say that F integrates Lipschitz curves if every Lipschitz curve $[a,b] \to F$, $a < b$, has a weak integral over $[a,b]$.

The following lemma plays, for Lipschitz curves, the role that the mean value inclusion plays for C^1 curves.

Lemma 5.22 (Lipschitz curves are continuous)**.** *Let γ be a Lipschitz curve on $[a,b]$ and $K = \operatorname{conv}(\{0\} \cup Q)$, where Q is the set of difference quotients. Then $\gamma(s) - \gamma(t) \in hK$ whenever $0 \le s - t \le h$. In particular γ is continuous.*

Proof. For $s \neq t$ we have $\gamma(s) - \gamma(t) = (s-t)\,q$ with $q \in Q$, and $(s-t)q \in hK$ since K is convex and contains 0. Continuity follows because the bounded set K is absorbed by every neighborhood of 0. □

5.6.1 Mackey complete spaces integrate Lipschitz curves

Theorem 5.23. *If F is Mackey complete, then F integrates Lipschitz curves.*

Proof. The argument is that of Theorem 5.11, with the difference quotients in place of the derivative. The set K of Lemma 5.22 is bounded by Lemma 5.5. By Lemmas 5.22 and 5.7,

$$R_{2^m}(\gamma) - R_{2^n}(\gamma) \in (b-a)\,\frac{b-a}{2^n}\,K \qquad (n \le m),$$

so the dyadic Riemann sums form a Mackey–Cauchy sequence. Its limit is the weak integral by Lemma 5.8. □

5.6.2 The converse

We start with an elementary observation, which allows us to work with a fast subsequence.

Lemma 5.24 (Subsequences)**.** *A Mackey–Cauchy sequence with a convergent subsequence converges, to the same limit.*

Proof. Let $u_{n_j} \to x$ and let W be a neighborhood of 0; choose a symmetric neighborhood V with $V + V \subseteq W$. For large k we have $r_k B \subseteq V$. Choosing j with $n_j \ge k$ and $u_{n_j} - x \in V$, we get

$$u_k - x = -(u_{n_j} - u_k) + (u_{n_j} - x) \in V + V \subseteq W.$$

□

The idea of the converse is the following. The increments of a Mackey–Cauchy sequence are placed on disjoint tents, which accumulate at $t = 1$; the tents are scaled so that the integral of the j-th one is the j-th increment. The resulting curve is Lipschitz, and its integral is then the sum of the increments, that is, the limit of the sequence up to a translation.

Definition 5.25 (Tents)**.** For $j \in \mathbb{N}$ let $l_j = 1 - 2^{-j}$ and $w_j = 2^{-(j+1)}$, so that the intervals $[l_j, l_j + w_j] = [l_j, l_{j+1}]$ tile $[0, 1)$. The j-th tent is the function

$$\tau_j(t) = \max\Big(0,\ \frac{w_j}{2} - \Big|t - l_j - \frac{w_j}{2}\Big|\Big),$$

supported on $[l_j, l_{j+1}]$, with slopes ± 1 and height $w_j/2$. For a sequence (e_j) in F we consider the curve $\gamma(t) = \sum_j \tau_j(t)\, e_j$, which is a finite sum for each t.

The slopes ± 1 are what makes γ Lipschitz, provided the vectors e_j stay in a bounded set; the fact that the tents become small near $t = 1$ is what allows the e_j to be large multiples of the increments.

Lemma 5.26 (Properties of the tents)**.** *Each τ_j is continuous, takes values in $[0, 1]$, satisfies $|\tau_j(s) - \tau_j(t)| \le |s - t|$, and $\int_0^1 \tau_j = w_j^2/4$. For each t, $\gamma(t) = \sum_{j<N} \tau_j(t)\, e_j$ for all large N.*

Proof. Elementary. The last claim holds because each t meets the support of at most finitely many tents. □

Theorem 5.27. *If F integrates Lipschitz curves, then F is Mackey complete.*

Proof. Let (u_n) be Mackey–Cauchy, with $u_l - u_k \in r_k B$ for $k \le l$. Let $K = \operatorname{conv}(\{0\} \cup B \cup -B)$, which is convex, symmetric and bounded by Lemma 5.5. Put $c_j = 2^{-(j+1)}$, so that $\sum_j c_j = 1$.

A fast subsequence. Since $r_k \to 0$, we can choose $n_0 < n_1 < \cdots$ with $|r_{n_j}| \le c_j w_j^2/4$. Let $d_j = u_{n_{j+1}} - u_{n_j}$ and $e_j = (4/w_j^2)\, d_j$. Then $e_j \in c_j K$.

The curve is Lipschitz. Let $\gamma = \sum_j \tau_j e_j$. A difference quotient of γ is a finite sum $\sum_j \frac{\tau_j(s)-\tau_j(t)}{s-t}\, e_j$ with coefficients of absolute value at most 1, hence it lies in $\sum_j c_j K \subseteq K$. So γ is a Lipschitz curve on $[0,1]$, and by hypothesis it has a weak integral y.

The integral is the limit. Since $\int_0^1 \tau_j e_j = d_j$, the partial sum $\gamma_N = \sum_{j<N} \tau_j e_j$ has weak integral $u_{n_N} - u_{n_0}$. On the other hand, $\gamma(t) - \gamma_N(t) = \sum_{j\geq N} \tau_j(t) e_j \in 2^{-N}K$ for every t. By Lemma 4.5, applied with W the closure of $2^{-N}K$, we get $y - (u_{n_N} - u_{n_0}) \in 2^{1-N}\,\overline{K}$, and this tends to 0 since K is bounded. Hence $u_{n_N} \to u_{n_0} + y$, and by Lemma 5.24 the whole sequence converges. □

Remark 5.28. The curve γ has corners at the ends and at the top of every tent, so it is not C^1 as soon as some $e_j \neq 0$. The argument therefore uses the full class of Lipschitz curves, and not only C^1 curves.

Corollary 5.29 (The two definitions agree)**.** *A locally convex space is Mackey complete if and only if it integrates Lipschitz curves.*

Proof. Theorems 5.23 and 5.27. □

In particular, the results of Chapter 5 stated under Mackey completeness hold verbatim with the definition of Glöckner and Neeb, and the references to [GN26] can be compared directly.

Chapter 6

Mapping spaces and the exponential law

Spaces of C^k maps are the models of manifolds of mappings and of the Lie groups of diffeomorphisms. In this chapter we give them the compact-open C^k topology and prove the results that make them usable [GN26, §1.7]: superposition operators are continuous and smooth, pullbacks are continuous linear maps, the differential is a continuous linear map, and the exponential law $C^{k,\ell}(U \times V, F) \cong C^k(U, C^\ell(V, F))$ holds, with an embedding in general and an isomorphism when V is locally compact. Chapter 17 explains why no better can be expected.

6.1 Superposition operators

Theorem 6.1 (Superposition operators are smooth [GN26, Prop. 1.7.1])**.** *Let K be a compact Hausdorff space, E, F locally convex spaces and $g\colon E \to F$ of class C^k on an open set $U \subseteq E$. With the compact-open topologies, the map $C(K,g)\colon C(K,U) \to C(K,F)$, $\gamma \mapsto g \circ \gamma$, is of class C^k, with $\mathrm{d}C(K,g)(\gamma,\eta) = (x \mapsto \mathrm{d}g(\gamma(x),\eta(x)))$.*

Proof. Continuity follows from the continuity of the evaluation $C(K,E) \times K \to E$. For the derivative, the difference quotients of g extend continuously to $t = 0$ (Glöckner's lemma), and the tube lemma over the compact space K gives convergence in the compact-open topology. The derivative is the superposition operator of $\mathrm{d}g$ on $C(K, E \times E) \cong C(K,E) \times C(K,E)$, and the theorem follows by induction on k. □

6.2 The compact-open C^k topology

Let E, F be locally convex spaces, $U \subseteq E$ open and $k \in \mathbb{N}$. In Lean, an element of $C^k(U,F)$ is a map $E \to F$ that is C^k on U and vanishes outside U; this does not change the mathematics and makes the family of differentials below point-separating.

Definition 6.2 (The compact-open C^k topology [GN26, Def. 1.7.10])**.** The compact-open C^k topology on $C^k(U,F)$ is the initial topology with respect to the linear maps $\mathrm{d}^{(j)}\colon C^k(U,F) \to C(U \times E^j, F)$, $\gamma \mapsto \mathrm{d}^{(j)}\gamma$, for $j \le k$, where $C(U \times E^j, F)$ carries the compact-open topology. It makes $C^k(U,F)$ a locally convex Hausdorff topological vector space.

Lemma 6.3 ([GN26, Lemma 1.7.15], embedding part)**.** *The map $C^k(U,F) \to \prod_{j\le k} C(U \times E^j, F)$, $\gamma \mapsto (\mathrm{d}^{(j)}\gamma)_{j\le k}$, is a linear topological embedding.*

Proof. This holds by definition of the topology; injectivity comes from $\mathrm{d}^{(0)}\gamma = \gamma$. □

Lemma 6.4 (Limits of differentiable maps)**.** *Let $U \subseteq E$ be open, $y \in E$, and let (g_i) be a net of maps $E \to F$ converging pointwise on U to g, such that each g_i has directional derivatives $D_i(z)$ in the direction y at all $z \in U$, with D_i continuous on U. Suppose that $D_i \to D$ in the compact-open sense on U (for every compact $K \subseteq U$ and open $O \supseteq D(K)$, eventually $D_i(K) \subseteq O$), with D continuous on U. Then g has directional derivative $D(x)$ in the direction y at every $x \in U$.*

Proof. Let W be a closed convex neighborhood of $D(x)$. For $|s| \le \delta$, $D(x+sy)$ lies in the interior of W, hence so does $D_i(x+sy)$ for i large. By the mean value inclusion, $t^{-1}(g_i(x+ty)-g_i(x)) \in W$ for $0 < |t| \le \delta$, and letting $i \to \infty$ the same holds for g, since W is closed. Glöckner and Neeb use instead their Exercise 1.1.13(b), whose hint goes through weak integrals; no completeness is needed here. □

Lemma 6.5 ([GN26, Lemma 1.7.15], closed image)**.** *The image of $C^k(U,F)$ in $\prod_{j\le k} C(U \times E^j, F)$ is closed.*

Proof. Let $h = (h_j)$ be in the closure of the image and $f := h_0$. By induction on j and Lemma 6.4, applied along the filter of γ with $(\mathrm{d}^{(j)}\gamma)_j$ close to h, the iterated directional derivatives of f exist and $\mathrm{d}^{(j)}f = h_j$. By Bastiani's characterization (Theorem 3.5), f is C^k. □

Proposition 6.6 (Linear operations [GN26, Prop. 1.7.11, Cor. 1.7.13, 1.7.14])**.** *The following maps are continuous and linear: the inclusion $C^l(U,F) \to C^k(U,F)$ for $k \le l$; the pullback $\gamma \mapsto \gamma\circ L$, $C^k(U,F) \to C^k(L^{-1}(U),F)$, along a continuous linear map $L\colon E_1 \to E$; the postcomposition $\gamma \mapsto g\circ\gamma$ with a continuous linear map $g\colon F \to G$. Moreover $C^k(U, F\times G) \cong C^k(U,F)\times C^k(U,G)$ as topological vector spaces.*

Proof. In each case $\mathrm{d}^{(j)}$ of the image is obtained from $\mathrm{d}^{(j)}$ of the argument by pre- or post-composition with a continuous map, or by pairing, and these operations are continuous in the compact-open topology (for pairing one restricts to compact subsets of $U \times E^j$, which are locally compact). □

Lemma 6.7 (The differentials of $\mathrm{d}\gamma$)**.** *Let $\gamma\colon U \to F$ be C^{k+1} and $j \le k$. For $x \in U$ and $a, v_1, \dots, v_j, w_1, \dots, w_j \in E$,*

$$\mathrm{d}^{(j)}(\mathrm{d}\gamma)((x,a);(v_1,w_1),\dots,(v_j,w_j)) = \mathrm{d}^{(j+1)}\gamma(x;a,v_1,\dots,v_j)+\sum_{i=1}^{j}\mathrm{d}^{(j)}\gamma(x;v_1,\dots,v_{i-1},w_i,v_{i+1},\dots,v_j).$$

Proof. Induction on j. In the inductive step, the first summand is differentiated in the direction $(v_{j+1},0)$ by definition of $\mathrm{d}^{(j+2)}\gamma$, and in the direction $(0,w_{j+1})$ by linearity of $\mathrm{d}^{(j+1)}\gamma(x;\cdot)$ in its first argument; Schwarz's theorem moves w_{j+1} to the last slot. The other summands depend only on x. □

Proposition 6.8 (The differential is continuous)**.** *The map $\mathrm{d}\colon C^{k+1}(U,F) \to C^k(U\times E,F)$, $\gamma \mapsto \mathrm{d}\gamma$, is continuous and linear, and the topology of $C^{k+1}(U,F)$ is initial with respect to $\gamma \mapsto \gamma \in C(U,F)$ and d.*

Proof. Continuity of d follows from Lemma 6.7. Conversely, $\mathrm{d}^{(j+1)}\gamma(x;y_0,\dots,y_j) = \mathrm{d}^{(j)}(\mathrm{d}\gamma)((x,y_0);(y_1,0),\dots,(y_j,0))$ (Lemma 3.4), so each $\mathrm{d}^{(j+1)}$ is $\mathrm{d}^{(j)}\circ\mathrm{d}$ followed by a precomposition. □

Theorem 6.9 (Superposition operators are continuous [GN26, Prop. 1.7.12, Cor. 1.7.13])**.** *Let Z, E, F be locally convex spaces, $U \subseteq Z$ open, $r \in \mathbb{N}$ and $f\colon Z \times E \to F$ of class C^r on $U \times E$. Then $f_*\colon C^r(U,E) \to C^r(U,F)$, $\gamma \mapsto f \circ (\mathrm{id}_U, \gamma)$, is continuous. In particular, for $g\colon E \to F$ of class C^k, the map $\gamma \mapsto g \circ \gamma$, $C^k(U,E) \to C^k(U,F)$, is continuous.*

Proof. Glöckner and Neeb use Faà di Bruno's formula. We argue by induction on r instead. For $r = 0$ this is the continuity of $\gamma \mapsto f \circ (\mathrm{id}, \gamma)$ on $C(U,E)$, which follows by restricting to compact subsets. By the chain rule, $\mathrm{d}(f_*\gamma)(x,v) = \mathrm{d}f((x,\gamma(x)),(v,\mathrm{d}\gamma(x,v)))$, that is, $\mathrm{d}(f_*\gamma) = (\mathrm{d}f)_*(\gamma \circ \mathrm{pr}_1, \mathrm{d}\gamma)$ with $(\gamma \circ \mathrm{pr}_1, \mathrm{d}\gamma) \in C^{r-1}(U \times Z, E \times E)$ depending continuously on γ (Propositions 6.6 and 6.8). The induction hypothesis and Proposition 6.8 conclude. □

Proposition 6.10 (Pullbacks [GN26, Prop. 1.7.11])**.** *Let E_1, E_2, F be locally convex spaces, $U \subseteq E_1$ and $V \subseteq E_2$ open, and $f\colon E_1 \to E_2$ of class C^r on U with $f(U) \subseteq V$. Then $f^*\colon C^r(V,F) \to C^r(U,F)$, $\gamma \mapsto \gamma \circ f$, is continuous and linear.*

Proof. Glöckner and Neeb use Faà di Bruno's formula; we argue by induction on r, as for Theorem 6.9. For $r = 0$, f^* is a precomposition. By the chain rule, $\mathrm{d}(\gamma \circ f)(x,v) = \mathrm{d}\gamma(f(x), \mathrm{d}f(x,v))$, so $\mathrm{d}(f^*\gamma) = g^*(\mathrm{d}\gamma)$ with $g(x,v) = (f(x), \mathrm{d}f(x,v))$ of class C^{r-1} on $U \times E_1$, and Proposition 6.8 and the induction hypothesis conclude. □

Theorem 6.11 (The space $C^\infty(U,F)$ [GN26, Def. 1.7.10, 1.7.11–1.7.15])**.** *The space $C^\infty(U,F)$ of smooth maps, with the initial topology with respect to all the $\mathrm{d}^{(j)}$, $j \in \mathbb{N}$ (equivalently, with respect to the inclusions into the $C^k(U,F)$), is a locally convex Hausdorff space; it embeds in $\prod_{j\in\mathbb{N}} C(U \times E^j, F)$ with closed image, and the results above (pullbacks, superposition operators, products, the differential) hold for $k = \infty$.*

Proof. Everything follows from the finite orders, since a map into $C^\infty(U,F)$ is continuous if and only if its compositions with the inclusions into the $C^k(U,F)$ are. For the closed image, the truncations of a point of the closure lie in the closed images of the $C^k(U,F)$, and their preimages all have the same underlying map. □

6.3 Smooth superposition operators and locality

Theorem 6.12 (Superposition operators are smooth)**.** *Let U be an open subset of a locally compact locally convex space Z, $O \subseteq E$ open and $h\colon E \to F$ smooth on O. If $g\colon X \to C^r(U,E)$ is C^k on an open set W and $g(w)(U) \subseteq O$ for $w \in W$, then $w \mapsto h \circ g(w)$ is C^k on W.*

Proof. By induction on r. The topology of $C^r(U,F)$ is initial with respect to the differentials $\mathrm{d}^{(j)}$, and $C^r(U,F)$ is closed in $\prod_j C(U \times Z^j, F)$, so it suffices to treat each $\mathrm{d}^{(j)}$. For $j = 0$, restriction to the compact subsets of U reduces the claim to superposition operators on $C(C,E)$ with C compact, which are smooth by Theorem 6.1. For $j \geq 1$, $\mathrm{d}(h \circ \gamma) = \mathrm{d}h \circ (\gamma \circ \mathrm{pr}_1, \mathrm{d}\gamma)$ is the superposition by the smooth map $\mathrm{d}h$ of an element of $C^{r-1}(U \times Z, E \times E)$ that depends continuously and linearly on γ, and the induction hypothesis applies on $U \times Z$. □

Remark 6.13. Glöckner and Neeb obtain the smoothness of the maps between spaces of C^ℓ maps that they need from the exponential law [GN26, Proposition 4.1.34]. The induction above avoids it. We do not claim it is new.

Proposition 6.14 (Locality [GN26, compare Lemma 1.7.43])**.**

1. *For an open cover (V_a) of U, the restrictions $C^r(U,F) \to \prod_a C^r(V_a,F)$ form a linear topological embedding with closed range. Hence being C^k, or converging, in $C^r(U,F)$ is a local property.*

2. *On a locally compact space Y, the restrictions $C(Y,F) \to \prod_C C(C,F)$ to the compact subsets form a closed embedding.*

3. *C^k maps into a product are the maps with C^k components, and a map into a closed subspace is C^k as soon as it is C^k into the ambient space.*

Proof. (a) The compact-open topology of $C(U \times Z^j, F)$ is initial with respect to the restrictions to the $V_a \times Z^j$, since every compact subset of U is covered by finitely many compact subsets of the V_a; a compatible family glues to a C^r map. (b) is the case of the cover by compact sets, and (c) follows from the definition of the differentials. □

6.4 $C^{k,\ell}$ maps and the exponential law

Let E_1, E_2, F be locally convex spaces, $U \subseteq E_1$ and $V \subseteq E_2$ open, and $k, \ell \in \mathbb{N}$.

Definition 6.15 ($C^{k,\ell}$ maps [GN26, Def. 1.7.17])**.** A map $f\colon U \times V \to F$ is $C^{k,\ell}$ if the mixed differentials

$$\mathrm{d}^{(i,j)} f(x,y;v_1,\dots,v_i;w_1,\dots,w_j) = (D_{(v_i,0)} \cdots D_{(v_1,0)} D_{(0,w_j)} \cdots D_{(0,w_1)} f)(x,y)$$

exist for $i \le k$, $j \le \ell$ and are continuous on $U \times V \times E_1^i \times E_2^j$. Equivalently (by Theorem 3.5), the maps $f(x,\cdot)$ are C^ℓ, the maps $x \mapsto \mathrm{d}^{(0,j)} f(x,y;w)$ are C^k, and the $\mathrm{d}^{(i,j)} f$ are continuous.

Lemma 6.16 (Mixed partial derivatives commute, cf. [GN26, Lemma 1.3.16])**.** *Let W be open and let g have directional derivatives A in the direction a and B in the direction b on W, with B continuous and with continuous directional derivatives M in the direction a. Then A has directional derivative M in the direction b.*

Proof. The difference quotients $r^{-1}(g(z+ra)-g(z))$ converge pointwise to A as $r \to 0$, and their derivatives in the direction b, $r^{-1}(B(z+ra)-B(z))$, converge to M uniformly on compact sets. Apply Lemma 6.4. Glöckner and Neeb differentiate a weak integral in a completion instead; no completeness is needed here. □

Proposition 6.17 ([GN26, Prop. 1.7.21, Cor. 1.7.22], the case used below)**.** *If f is $C^{k,\ell}$, $i \le k$, $x \in U$ and $v \in E_1^i$, then $y \mapsto \mathrm{d}^{(i,0)} f(x,y;v)$ is C^ℓ on V with j-th differential $\mathrm{d}^{(i,j)} f(x,\cdot;v;\cdot)$.*

Proof. By induction on i, Lemma 6.16 moves one derivative in the second variable past the derivatives in the first variable. □

Proposition 6.18 (Schwarz theorem for $C^{k,\ell}$ maps [GN26, Prop. 1.7.21])**.** *Let f be $C^{k,\ell}$, $i \le k$, $j \le \ell$, $v \in E_1^i$, $w \in E_2^j$ and π a permutation of $\{1,\dots,i+j\}$. The iterated directional derivative of f along $(v_1,0),\dots,(v_i,0),(0,w_1),\dots,(0,w_j)$, taken in the order prescribed by π, exists at every point of $U \times V$ and equals $\mathrm{d}^{(i,j)} f(x,y;v;w)$, which is symmetric in v and in w.*

Proof. By induction on the number of directions: the last one is either of the form $(v,0)$, handled by the definition of $\mathrm{d}^{(i,j)} f$, or $(0,w)$, handled by Proposition 6.17. The symmetry in v is the symmetry of the differentials of $x \mapsto \mathrm{d}^{(0,j)} f(x,y;w)$; the symmetry in w follows from Proposition 6.17. □

Definition 6.19 (The space $C^{k,\ell}(U \times V, F)$ [GN26, Def. 1.7.17])**.** $C^{k,\ell}(U \times V, F)$ carries the initial topology with respect to the linear maps $\mathrm{d}^{(i,j)} \colon C^{k,\ell}(U \times V, F) \to C(U \times V \times E_1^i \times E_2^j, F)$, $i \le k$, $j \le \ell$ (compact-open topology on the right). It is a locally convex Hausdorff space.

Proposition 6.20 ([GN26, Prop. 1.7.33(a)])**.** *Every $C^{k+\ell}$ map on $U \times V$ is $C^{k,\ell}$, with $\mathrm{d}^{(i,j)} f(x, y; v; w) = \mathrm{d}^{(j+i)} f((x, y); (0, w_1), \dots, (0, w_j), (v_1, 0), \dots, (v_i, 0))$, and the inclusion $C^{k+\ell}(U \times V, F) \to C^{k,\ell}(U \times V, F)$ is continuous and linear.*

Proof. The formula is an identity between iterated limits; the right hand side is continuous and depends continuously on f. □

Theorem 6.21 (Exponential law [GN26, Thm. 1.7.30])**.** *For $f \in C^{k,\ell}(U \times V, F)$, the map $f^\vee \colon U \to C^\ell(V, F)$, $x \mapsto f(x, \cdot)$, is C^k, with $\mathrm{d}^{(j)}(\mathrm{d}^{(i)} f^\vee(x; v))(y; w) = \mathrm{d}^{(i,j)} f(x, y; v; w)$, and $\Phi \colon C^{k,\ell}(U \times V, F) \to C^k(U, C^\ell(V, F))$, $f \mapsto f^\vee$, is linear and a topological embedding. If E_2 is locally compact (for nonempty open V, this is the same as V being locally compact), Φ is an isomorphism of topological vector spaces, with inverse $g \mapsto ((x, y) \mapsto g(x)(y))$.*

Proof. The candidate for $\mathrm{d}^{(i)} f^\vee(x; v)$ is $y \mapsto \mathrm{d}^{(i,0)} f(x, y; v)$, which lies in $C^\ell(V, F)$ by Proposition 6.17. The difference quotients of $x \mapsto \mathrm{d}^{(i,j)} f(x, \cdot; v; \cdot)$ converge uniformly on compact sets, so the derivative exists in $C^\ell(V, F)$, and Bastiani's characterization shows that $f^\vee$ is C^k. Both Φ and the $\mathrm{d}^{(i,j)}$ are recovered from each other by evaluations at points of compact sets, which are jointly continuous; hence Φ is an embedding. If E_2 is locally compact, $V \times E_2^j$ is locally compact, so for $g \in C^k(U, C^\ell(V, F))$ the map $(x, y, v, w) \mapsto \mathrm{d}^{(j)}(\mathrm{d}^{(i)} g(x; v))(y; w)$ is continuous; it is the mixed differential $\mathrm{d}^{(i,j)}$ of $(x, y) \mapsto g(x)(y)$, which is therefore $C^{k,\ell}$. Glöckner and Neeb use the evaluation map and a chain rule for $C^{r,s}$ maps (with Faà di Bruno's formula) instead. □

Proposition 6.22 ([GN26, Prop. 1.7.33(b)])**.** *Every $C^{k,k}$ map on $U \times V$ is C^k, and the inclusion $C^{k,k}(U \times V, F) \to C^k(U \times V, F)$ is continuous and linear.*

Proof. Induction on k. By the rule on partial differentials, a $C^{k+1,k+1}$ map f is C^1 with $\mathrm{d}f((x, y), (a, b)) = \mathrm{d}^{(1,0)} f(x, y; a) + \mathrm{d}^{(0,1)} f(x, y; b)$. As a function of $((x, a), (y, b))$, the right hand side is $C^{k,k}$ on $(U \times E_1) \times (V \times E_2)$: its mixed differentials are explicit sums of mixed differentials of f, by Lemma 6.7 and Proposition 6.17. The induction hypothesis gives that $\mathrm{d}f$ is C^k, and also, with Proposition 6.8, the continuity of the inclusion. Glöckner and Neeb prove instead the explicit formula of Proposition 6.23 for $\mathrm{d}^{(m)} f$. □

Proposition 6.23 ([GN26, Prop. 1.7.33(b), (1.108)])**.** *If f is $C^{m,m}$, $(x, y) \in U \times V$ and $w \in (E_1 \times E_2)^m$, then*

$$\mathrm{d}^{(m)} f(x, y; w) = \sum_{I \subseteq \{1, \dots, m\}} \mathrm{d}^{(|I|, m-|I|)} f(x, y; (w_{i,1})_{i \in I}; (w_{j,2})_{j \notin I}),$$

with the indices in increasing order.

Proof. By Proposition 6.22, $\mathrm{d}^{(m)} f(x, y; \cdot)$ is multilinear; expand it along $w_a = (w_{a,1}, 0) + (0, w_{a,2})$ and apply Proposition 6.18 to each term. □

Corollary 6.24 ($C^\infty = C^{\infty,\infty}$ [GN26, Cor. 1.7.34])**.** *A map on $U \times V$ is smooth if and only if it is $C^{k,\ell}$ for all k, ℓ, and $C^\infty(U \times V, F) = C^{\infty,\infty}(U \times V, F)$ as locally convex spaces.*

Proof. Combine Propositions 6.20 and 6.22. □

Lemma 6.25 (C^k maps into $C^\infty(V, F)$)**.** *A map $g \colon W \to C^\infty(V, F)$ is C^k if and only if its compositions with the inclusions $C^\infty(V, F) \to C^\ell(V, F)$ are C^k, for all ℓ.*

Proof. Induction on k. The derivatives of the compositions have the same underlying maps, hence define a candidate in $C^\infty(V, F)$, and the topology of $C^\infty(V, F)$ is initial with respect to the inclusions. □

Theorem 6.26 (Exponential law for smooth maps [GN26, Cor. 1.7.35, Rem. 1.7.31(b)])**.** *For $f \in C^\infty(U \times V, F)$, $f^\vee \colon U \to C^\infty(V, F)$ is smooth, and $\Phi \colon C^\infty(U \times V, F) \to C^\infty(U, C^\infty(V, F))$, $f \mapsto f^\vee$, is linear and a topological embedding. If E_2 is locally compact, Φ is an isomorphism of topological vector spaces, and a map $g \colon U \to C^\infty(V, F)$ is smooth if and only if $(x, y) \mapsto g(x)(y)$ is smooth.*

Proof. Smoothness of $f^\vee$ follows from Theorem 6.21 and Lemma 6.25. The embedding is proved as in Theorem 6.21, using that the topology of $C^\infty(U \times V, F)$ is the $C^{\infty,\infty}$ topology (Corollary 6.24). □

Chapter 7

C^k maps on sets with dense interior

Cubes $[0,1]^n$, half-spaces and their open subsets are not open, but they are the domains of the charts of manifolds with boundary or with corners, and of the curves on compact intervals. Glöckner and Neeb extend the calculus to *locally convex subsets with dense interior* [GN26, §1.4]: a map is C^1 on such a set U if it is C^1 on the interior U^0 and its differential extends continuously to $U \times E$. In this chapter we prove that the first-order calculus of Chapter 2, the higher differentials of Chapter 3, and the $C^{k,\ell}$ maps and the exponential law of Chapter 6 all survive on such sets. Manifolds with rough boundary, whose charts take values in these sets [GN26, Definition 3.5.1], are built on this chapter in lc-manifolds.

As everywhere else, no completeness is assumed: integrals are weak integrals. In a few places the proofs depart from those of [GN26]; the remarks say where.

7.1 Locally convex subsets and C^k maps

Definition 7.1 (Locally convex subsets with dense interior [GN26, Definitions 1.4.1 and 1.4.4])**.** A subset U of a locally convex space E is *locally convex* if every point of U has a convex neighbourhood in U, and it has *dense interior* if $U \subseteq \overline{U^0}$. A map $f\colon U \to F$ on such a set is C^0 if it is continuous. It is C^{k+1} if it is continuous, $f|_{U^0}$ has all directional derivatives, and the differential $\mathrm{d}(f|_{U^0})\colon U^0 \times E \to F$ extends to a map $\mathrm{d}f\colon U \times E \to F$ which is C^k on $U \times E$.

The definition is recursive, as Definition 2.3; Glöckner and Neeb state it through the iterated differentials, and the two forms agree by Theorem 7.6 below.

Proposition 7.2 (Basic properties)**.** *Let $U \subseteq E$ be a locally convex subset with dense interior.*

1. *On open sets, C^k in the sense of Definition 7.1 is C^k in the sense of Definition 2.3.*
2. *The continuous extension of $\mathrm{d}(f|_{U^0})$ is unique.*
3. *f is C^{k+1} as soon as $\mathrm{d}(f|_{U^0})$ has some C^k extension to $U \times E$, and then $\mathrm{d}f$ is that extension.*
4. *If f is C^1, then $\mathrm{d}f(x,\cdot)$ is continuous and linear for every $x \in U$ [GN26, Lemma 1.4.7].*

Proof. (b) holds because U^0 is dense in U; (c) is a reformulation of the definition, and all the rules below are proved from it. (d): $\mathrm{d}f(x,\cdot)$ is a pointwise limit of the linear maps $\mathrm{d}f(y,\cdot)$, $y \in U^0$, so it is linear, and it is continuous by definition. □

7.2 First-order calculus

Theorem 7.3 (Mean value theorem and difference quotients [GN26, Proposition 1.4.8, Lemma 1.4.9])**.** *Let $f\colon U \to F$ be C^1.*

1. *If the segment $[x, x+h]$ lies in U, then $\lambda(f(x+h) - f(x)) = \int_0^1 \lambda(\mathrm{d}f(x+th, h))\,dt$ for every continuous linear functional λ on F.*
2. *The difference quotients $t^{-1}(f(x+th) - f(x))$ extend continuously to $t = 0$, with value $\mathrm{d}f(x, h)$.*

Proof. For a convex U we shrink the segment towards an interior point, where Proposition 2.10 applies, and pass to the limit in a parameter-dependent integral; this gives (a) for convex sets, and (b). For a general locally convex U, the function $t \mapsto \lambda(f(x+th))$ has right derivative $\lambda(\mathrm{d}f(x+th, h))$ on $[0, 1)$ by (b), and (a) follows from the fundamental theorem of calculus for real functions. □

Remark 7.4. Glöckner and Neeb prove the mean value theorem first on convex sets, with an integral in a completion of F, and then split a segment into finitely many pieces lying in convex neighbourhoods. The argument above avoids the completion. We do not claim it is new.

Theorem 7.5 (Rules of calculus [GN26, Propositions 1.4.10 and 1.4.11])**.** *On locally convex subsets with dense interior:*

1. ***Chain rule.*** *If $f\colon U \to V$ and $g\colon V \to G$ are C^k, so is $g \circ f$, with $\mathrm{d}(g \circ f)(x, h) = \mathrm{d}g(f(x), \mathrm{d}f(x, h))$.*
2. *Pairs, products of maps, compositions with continuous linear maps, sums and continuous bilinear images of C^k maps are C^k; restrictions of C^k maps on open sets are C^k.*
3. *C^k is inherited by locally convex subsets with dense interior, and it is a local property.*
4. ***Partial differentials.*** *A map on $U_1 \times U_2$ is C^1 if and only if its partial differentials exist and extend continuously, and then $\mathrm{d}f((x, y), (h, k)) = \mathrm{d}_1 f((x, y), h) + \mathrm{d}_2 f((x, y), k)$.*

Proof. (a) The candidate differential $\mathrm{d}g(f(x), \mathrm{d}f(x, h))$ is continuous, and on U^0 it is the derivative of $g \circ f$ by Theorem 7.3(b) applied to g along $f(x+th)$; conclude with Proposition 7.2(c), by induction on k. The chain rule needs only V to be locally convex with dense interior, and only density of U^0 in U. (b)–(d) follow from (a) and Proposition 7.2(c) as in the open case. □

Items (b) and (c) are what the transition maps of manifolds with rough boundary need [GN26, Remark 3.5.2].

7.3 Higher differentials, curves and subspaces

Theorem 7.6 (Higher differentials [GN26, Lemma 1.4.5])**.** *A map $f\colon U \to F$ is C^k if and only if it is continuous, $f|_{U^0}$ is C^k, and the iterated differentials $\mathrm{d}^{(j)}(f|_{U^0})$ extend continuously to $U \times E^j$ for $j \le k$. The extensions are unique.*

Proof. Necessity by induction on k, from the definition. For sufficiency, [GN26] uses the version of [GN26, Lemma 1.3.12] for sets with dense interior [GN26, Remark 1.4.12]. We argue by induction on k instead, with the explicit formula of Theorem 3.5 for the differentials of $\mathrm{d}(f|_{U^0})$ on the open set $U^0 \times E$, whose right-hand side extends continuously. We do not claim this argument is new. □

Proposition 7.7 (Curves and subspaces [GN26, Lemmas 1.4.13 and 1.4.16])**.**

1. *On a non-degenerate interval $I \subseteq \mathbb{R}$, the C^k maps in the sense of Definition 7.1 are the C^k curves, with $\mathrm{d}\gamma(s,r) = r\gamma'(s)$.*
2. *Let $F_0 \subseteq F$ be a sequentially closed vector subspace. A map $U \to F_0$ is C^k into F if and only if it is C^k into F_0.*

Proof. (a) At interior points both notions are the derivative; at the end points both ask for a continuous extension. (b) Every point of U is the limit of a sequence in U^0, so the continuous extensions of the differentials take values in F_0 when those on U^0 do. □

7.4 $C^{k,\ell}$ maps and the exponential law

Definition 7.8 ($C^{k,\ell}$ maps on sets with dense interior [GN26, Definition 1.7.17])**.** Let $U \subseteq E_1$ and $V \subseteq E_2$ be locally convex subsets with dense interior. A map $f\colon U \times V \to F$ is $C^{k,\ell}$ if it is continuous, it is $C^{k,\ell}$ on $U^0 \times V^0$ (Definition 6.15), and the mixed differentials $\mathrm{d}^{(i,j)}(f|_{U^0\times V^0})$, $i \le k$, $j \le \ell$, extend to continuous maps on $U \times V \times E_1^i \times E_2^j$. For $k, \ell \in \mathbb{N}_0 \cup \{\infty\}$, f is $C^{k,\ell}$ if it is $C^{i,j}$ for all finite $i \le k$, $j \le \ell$. On open sets this is Definition 6.15.

Theorem 7.9 (Properties of $C^{k,\ell}$ maps [GN26, Lemma 1.7.18, Remark 1.7.19(b), Propositions 1.7.20, 1.7.26, 1.7.28, 1.7.33(a)])**.** *Let $f\colon U \times V \to F$ be $C^{k,\ell}$.*

1. *For every $x \in U$, also on the boundary, $f(x,\cdot)$ is C^ℓ with $\mathrm{d}^{(j)}(f(x,\cdot)) = \mathrm{d}^{(0,j)}f(x,\cdot)$; for $y \in V$ and $w \in E_2^j$, $x \mapsto \mathrm{d}^{(0,j)}f(x,y,w)$ is C^k.*
2. ***Recursion.*** *f is $C^{k+1,\ell}$ if and only if it is $C^{0,\ell}$ and $((x,a),y) \mapsto \mathrm{d}^{(1,0)}f(x,y,a)$ is $C^{k,\ell}$ on $(U \times E_1) \times V$; and similarly in the second variable.*
3. *A map into a product is $C^{k,\ell}$ if and only if its components are.*
4. ***Chain rules.*** *$(x,y) \mapsto g(f_1(x), f_2(y))$ is $C^{r,s}$ if g is $C^{r,s}$, f_1 is C^r and f_2 is C^s; and $g \circ f$ is $C^{r,s}$ if f is $C^{r,s}$ and g is C^{r+s}.*
5. *$C^{k+\ell}$ maps on $U \times V$ are $C^{k,\ell}$.*

Proof. (a) Pass to the limit from U^0 with a parametrized form of the mean value inclusion: a family of maps with continuous directional derivatives keeps them at accumulation points of the parameters (the role of [GN26, Lemma 1.4.18]); conclude with Theorem 7.6. (b) From explicit formulas for the mixed differentials of the partial differentials, and the interchange of a derivative in the first variable with derivatives in the second. (c)–(e) follow from (b) by induction, with the chain rule of Theorem 7.5. □

Remark 7.10. Glöckner and Neeb prove the chain rules by computing the mixed differentials of the composite with Faà di Bruno's formula [GN26, (1.91)–(1.97)]. The recursive characterization (b) avoids those formulas. We do not claim this argument is new.

Theorem 7.11 (Exponential law on sets with dense interior [GN26, Theorem 1.7.30])**.** *Give $C^k(V,F)$ and $C^{k,\ell}(U \times V, F)$ the compact-open C^k and $C^{k,\ell}$ topologies, as in Chapter 6. For $f \in C^{k,\ell}(U \times V, F)$, the map $f^\vee\colon U \to C^\ell(V,F)$, $f^\vee(x) = f(x,\cdot)$, is C^k, with*

$$\mathrm{d}^{(j)}(\mathrm{d}^{(i)}(f^\vee)(x,v))(y,w) = \mathrm{d}^{(i,j)}f(x,y,v,w) \qquad \textit{[GN26, (1.98)]},$$

and $\Phi\colon f \mapsto f^\vee$ is a linear topological embedding $C^{k,\ell}(U \times V, F) \to C^k(U, C^\ell(V,F))$. If V is locally compact, Φ is an isomorphism of topological vector spaces. The same holds for $k = \ell = \infty$.

Proof. At interior points the derivative of $f^\vee$ is $(\mathrm{d}^{(1,0)}f)^\vee$: the difference quotients converge in $C^\ell(V,F)$, uniformly on compact subsets of $V\times E_2^j$, by the mean value inclusion with a parameter ranging over all of V. The induction then runs through the recursion of Theorem 7.9(b). For $k=\ell=\infty$, $C^\infty(V,F)$ is the closed subspace of compatible families in $\prod_j C^j(V,F)$, and the law follows from the finite orders. □

Remark 7.12. Glöckner and Neeb differentiate $f^\vee$ through the continuous extension of difference quotients of $C^{1,0}$ maps [GN26, Lemma 1.7.6] and an induction on the order of the differentials [GN26, (1.99)]. Their alternative hypothesis (b) on $k_{\mathbb{R}}$-spaces is also formalized (`curryEquivOfKRSpace`); first countable spaces are $k_{\mathbb{R}}$-spaces, so the law holds for Fréchet spaces.

Chapter 8

Comparison with Fréchet calculus

On normed spaces two calculi are available: the classical Fréchet calculus, in which the derivative is required to depend continuously on the point in the operator norm, and the calculus of Michal and Bastiani, which only asks for joint continuity of $(x, v) \mapsto \mathrm{d}f(x)(v)$. The second notion is weaker; Fréchet C^k maps are Michal–Bastiani C^k (Proposition 2.19). In this chapter we prove the converse, up to the loss of one derivative. As a consequence both notions of smoothness agree, so that for smooth maps between normed spaces one may use whichever calculus is more convenient.

In the second part of the chapter we apply this comparison to maps with bounded below derivative, and show that such maps reflect smoothness. Throughout, E and F are real normed spaces; completeness is assumed only where it is stated.

8.1 From Michal–Bastiani to Fréchet

Theorem 8.1 (Michal–Bastiani C^{k+1} implies Fréchet C^k)**.** *If f is C^{k+1} on U in the sense of Michal–Bastiani, then f is Fréchet C^k on U.*

The extra derivative is what converts joint continuity into continuity in the operator norm: in the proof below, joint continuity of the next derivative, together with the mean value inclusion, shows that the derivatives are locally Lipschitz in the operator norm.

Proof. We prove, by induction on k and for all m simultaneously, the following statement: *if $g\colon U \to \mathcal{L}^m(E;F)$ is such that $\hat{g}(x, v_1, \dots, v_m) = g(x)(v_1, \dots, v_m)$ is C^{k+1} in the sense of Michal–Bastiani, then g is Fréchet C^k.* Here $\mathcal{L}^m(E;F)$ is the space of continuous m-linear maps with the operator norm. For $m = 0$ this is the theorem.

The candidate for the derivative of g at y is the $(m+1)$-linear map $B_y(w, v) = \mathrm{d}\hat{g}((y, v), (w, 0))$, the derivative of $\hat{g}$ in a base direction; it is multilinear by Theorem 2.12. Since $\mathrm{d}\hat{g}$ is jointly continuous and vanishes in the direction 0, the maps $B_{y'}$ are uniformly bounded in operator norm for y' near y. With the mean value inclusion (Lemma 2.8), this shows that g is locally Lipschitz into $\mathcal{L}^m(E;F)$; in particular it is continuous, which is the case $k = 0$.

For the inductive step, suppose that $\hat{g}$ is C^{k+2}. Then $(y, w, v) \mapsto B_y(w, v)$ is C^{k+1}, so by the case $(k, m+1)$ the map $y \mapsto B_y$ is Fréchet C^k, and in particular continuous in operator norm. The mean value inclusion then gives

$$\|g(y + w) - g(y) - B_y(w, \cdot)\| = o(\|w\|),$$

so g is Fréchet differentiable with derivative $y \mapsto B_y$, and g is Fréchet C^{k+1}. □

Remark 8.2. This is [GN26, 2.3.17(b)]. The proof in [GN26] goes through the compact-open topology on $\mathcal{L}(E, F)$ and the exponential law ([GN26, 1.7.30]). The proof above stays in normed spaces: the derivatives are tracked as continuous multilinear maps with the operator norm, which is closer to the existing Fréchet calculus of Mathlib. The price is the auxiliary statement for all m, which has to be proved simultaneously.

Since the loss of one derivative disappears when $k = \infty$, we obtain the following.

Corollary 8.3 (Smooth maps agree)**.** *A map between open subsets of normed spaces is smooth in the sense of Michal–Bastiani if and only if it is smooth in the sense of Fréchet.*

Proof. Theorem 8.1 and Proposition 2.19. □

8.2 Operator-valued maps on locally convex spaces

Theorem 8.4 (Operator-valued maps)**.** *Let E be a locally convex space, F_1, F_2 normed spaces, $U \subseteq E$ open and $g\colon U \to \mathcal{L}(F_1, F_2)$. If $\hat{g}(y, \xi) = g(y)\xi$ is C^{k+1} on $U \times F_1$ in the sense of Michal–Bastiani, then g is C^k for the operator norm on $\mathcal{L}(F_1, F_2)$.*

Proof. Let $D(y, \xi, w) = \mathrm{d}\hat{g}((y, \xi), (w, 0))$; it is linear in ξ and in w, and continuous. Continuity at $(y_0, 0, 0)$ gives a convex neighborhood N of y_0, a neighborhood W of 0 and C with $\|D(y, \xi, w)\| \leq C\|\xi\|$ for $y \in N$, $w \in W$. If $y - y_0 \in rW$, the mean value inequality along the segment $[y_0, y] \subseteq N$ gives $\|g(y) - g(y_0)\| \leq rC$: g is continuous. The operator $h(y, w) = D(y, \cdot, w)$ is the directional derivative of g for the operator norm, by the mean value inequality applied to $s \mapsto g(y + sw)\xi - s\,h(y, w)\xi$ and the continuity of h, which is the case $k = 0$ applied to h on $U \times E$. The induction runs on the domain $E \times E$ of the derivative. □

The statement is meaningless if F_1 is not normable: no vector topology on $\mathcal{L}(F_1, F_2)$ makes the evaluation continuous. Theorem 8.1 is the case where E is normed; there the multilinear version is needed for the induction.

8.3 Maps with bounded below derivative

In this section we consider a Banach space E (complete), a normed space F (complete or not), and a map $j\colon U \to F$, smooth on an open set $U \subseteq E$, with derivative bounded below on U: there is $m > 0$ with

$$m\|v\| \leq \|\mathrm{d}j(x)(v)\| \qquad (x \in U,\ v \in E).$$

By Corollary 8.3 it makes no difference in which sense j is smooth. No complement of the range of $\mathrm{d}j(x)$ is assumed. The simplest example is a continuous linear map $j\colon E \to F$ with $m\|v\| \leq \|j(v)\|$, for instance an isometric embedding; in that case $\mathrm{d}j(x) = j$ for every x.

Our aim is to show that such a j reflects smoothness: a continuous map γ into U is of class C^k as soon as $j \circ \gamma$ is. The test maps γ are defined on open subsets of arbitrary locally convex spaces. We first prove a linear version, in which the unknown appears linearly.

Lemma 8.5 (Linear bootstrap)**.** *Let W be an open subset of a locally convex space H, $\gamma\colon W \to U$ and $w\colon W \to F$ of class C^k, and $z\colon W \to E$ continuous with $\mathrm{d}j(\gamma q)(zq) = w(q)$ for $q \in W$. Then z is of class C^k.*

Proof. By induction on k. For the inductive step, fix $q \in W$ and a direction h, and write $A_t = \mathrm{d}j(\gamma(q+th))$. From the equation,

$$A_0\Big(\frac{z(q+th)-z(q)}{t}\Big) = \frac{w(q+th)-w(q)}{t} - \frac{A_t - A_0}{t}\, z(q+th).$$

Since j is Fréchet C^2 (Theorem 8.1), $t \mapsto A_t$ is differentiable in the operator norm, so the right-hand side converges. As A_0 is bounded below and E is complete, the difference quotients of z are Cauchy, hence converge to some D with $A_0 D = \mathrm{d}w(q)(h) - \mathrm{d}^2 j(\gamma q)(\mathrm{d}\gamma(q)(h), zq)$.

The derivative $\mathrm{d}z$ is continuous, because solutions of $A(p)x(p) = r(p)$ are continuous when A is continuous in the operator norm and uniformly bounded below. Moreover, it satisfies an equation of the same type on $W \times H$, with a right-hand side of class C^k by the inductive hypothesis applied to z. So $\mathrm{d}z$ is C^k by the inductive hypothesis again, and z is C^{k+1}. □

Theorem 8.6 (Bounded below derivatives reflect smoothness)**.** *Let W be an open subset of a locally convex space and $\gamma\colon W \to U$ continuous. If $j \circ \gamma$ is of class C^k, then so is γ.*

In the linear example above, the theorem says that a continuous map into E is C^k as soon as it is C^k when regarded, through j, as a map into the larger space F.

Proof. By induction on k. Fix $q \in W$, a direction h, and write $A = \mathrm{d}j(\gamma q)$, $d_t = \gamma(q+th) - \gamma(q)$. Since j is strictly differentiable at γq, for any $c > 0$ we have

$$\|j(\gamma(q+th)) - j(\gamma q) - A d_t\| \le c\|d_t\|$$

for small t. With $c \le m/2$ the lower bound gives $\|d_t/t\| \le (2/m)\|(j(\gamma(q+th)) - j(\gamma q))/t\|$, so $A(d_t/t) \to \mathrm{d}(j\circ\gamma)(q)(h)$, and as in Lemma 8.5 the quotients d_t/t converge. The derivative $\mathrm{d}\gamma$ then solves

$$\mathrm{d}j(\gamma q)(\mathrm{d}\gamma(q)(h)) = \mathrm{d}(j\circ\gamma)(q)(h),$$

and Lemma 8.5 applies. □

Remark 8.7. A word of caution on the scope of this result. The proof uses that $x \mapsto \mathrm{d}j(x)$ is continuous in the operator norm, to keep the lower bound near a point. The Michal–Bastiani definition only gives the joint continuity of $(x, v) \mapsto \mathrm{d}j(x)(v)$; on normed spaces the operator norm continuity is recovered through Theorem 8.1. For locally convex model spaces, Section 8.4 proves the statement under a locally uniform lower bound, which is automatic in normed spaces; with only a pointwise lower bound it is open.

8.4 Uniformly injective derivatives on locally convex spaces

We now let E, F be Hausdorff locally convex spaces, E sequentially complete (Definition 5.1), and $j\colon U \to F$ of class C^2 on an open set $U \subseteq E$. For a single continuous linear map $L\colon E \to F$ the analogue of being bounded below is being a topological embedding: for every continuous seminorm p on E there are a continuous seminorm q on F and $C \ge 0$ with $p(w) \le C\, q(Lw)$. The proof of Theorem 8.6 needs this near each point, uniformly, and also for convex combinations of derivatives. We found no source for the results of this section; they were written for this library by adapting the Banach proof, and we do not claim they are new.

Definition 8.8 (Uniformly injective derivatives)**.** The derivatives of j are *uniformly injective on* $V \subseteq U$ if for every continuous seminorm p on E there are a continuous seminorm q on F and $C \geq 0$ with

$$p(w) \leq C\, q(z) \qquad \text{for all } w \in E \text{ and } z \in \overline{\operatorname{conv}}\,\{\mathrm{d}j(y)(w) : y \in V\}.$$

We say that this holds *near* x_0 if it holds on some neighborhood of x_0.

Equivalently, $p(w) \leq C\, q(\sum_i \lambda_i \mathrm{d}j(y_i)(w))$ for all finite convex combinations with $y_i \in V$. Taking a single point, each $\mathrm{d}j(y)$, $y \in V$, is a topological embedding with constants independent of y. The hypothesis does not follow from each $\mathrm{d}j(x)$ being an embedding: in the Fréchet space of rapidly decreasing sequences there is a smooth family of linear automorphisms $L(t)$ with $L(0) = \mathrm{id}$ whose inverses are not equicontinuous near $t = 0$ (Example 16.4).

Lemma 8.9 (The normed case)**.** *If E, F are normed and $\mathrm{d}j(x_0)$ is bounded below, then the derivatives of j are uniformly injective near x_0.*

Proof. Let $m\|v\| \leq \|\mathrm{d}j(x_0)v\|$. By Theorem 8.1, $x \mapsto \mathrm{d}j(x)$ is continuous in the operator norm, so $\|\mathrm{d}j(y) - \mathrm{d}j(x_0)\| < m/2$ on a neighborhood V of x_0. The closed ball of radius $\frac{m}{2}\|w\|$ around $\mathrm{d}j(x_0)w$ is closed and convex and contains every $\mathrm{d}j(y)w$, $y \in V$, hence every z in their closed convex hull, and then $\|z\| \geq \frac{m}{2}\|w\|$. A continuous seminorm satisfies $p \leq c\|\cdot\|$. □

Lemma 8.10 (Difference quotients)**.** *Assume the derivatives of j are uniformly injective near $x \in U$. If $d_i \to 0$, $\tau_i \neq 0$ and $\tau_i^{-1}(j(x + d_i) - j(x)) \to R$ along a countably generated filter, then $\tau_i^{-1} d_i \to D$ with $\mathrm{d}j(x)D = R$.*

Proof. Write $g_i = \tau_i^{-1} d_i$ and $\Delta_i = \tau_i^{-1}(j(x+d_i)-j(x))$. By the mean value inclusion (Lemma 2.8), Δ_i lies in the closed convex hull of $\{\mathrm{d}j(y)(g_i) : y \in [x, x + d_i]\}$, and the segment lies in V for large i; so $p(g_i) \leq Cq(\Delta_i)$ is eventually bounded, for each continuous seminorm p.

Joint continuity of $\mathrm{d}^2 j$ at $(x, 0, 0)$ and homogeneity give, for each continuous seminorm q on F, a continuous seminorm r on E with $q(\mathrm{d}^2 j(z)(u, v)) \leq r(u)r(v)$ for z near x. The mean value inclusion, twice, gives $q(\Delta_i - \mathrm{d}j(x)g_i) \leq r(d_i)\, r(g_i) \to 0$. Hence $\mathrm{d}j(x)g_i \to R$. As $\mathrm{d}j(x)$ is an embedding, (g_{i_n}) is Cauchy along a sequence i_n tending to the filter, and converges to some D because E is sequentially complete; then $\mathrm{d}j(x)D = R$, and $p(g_i - D) \leq Cq(\mathrm{d}j(x)g_i - R) \to 0$. □

Lemma 8.11 (Linear bootstrap, locally convex case)**.** *Assume the derivatives of j are uniformly injective near every point of U and j is C^{k+1}. Let W be an open subset of a locally convex space, $\gamma\colon W \to U$ and $w\colon W \to F$ of class C^k, and $z\colon W \to E$ continuous with $\mathrm{d}j(\gamma q)(zq) = w(q)$. Then z is of class C^k.*

Proof. As in Lemma 8.5. The second order term $t^{-1}(\mathrm{d}j(\gamma(q+th))(z(q+th)) - \mathrm{d}j(\gamma q)(z(q+th)))$ converges to $\mathrm{d}^2 j(\gamma q)(\mathrm{d}\gamma(q)(h), zq)$ by Glöckner's lemma (Lemma 2.11) for $\mathrm{d}j$, and the limit of the quotients of z is produced as in Lemma 8.10. Continuity of $\mathrm{d}z$: if $\mathrm{d}j(\gamma a)(za) = r(a)$, then $p(za' - za) \leq Cq(r(a') - \mathrm{d}j(\gamma a')(za)) \to 0$ by joint continuity of $\mathrm{d}j$. □

Theorem 8.12 (Uniformly injective derivatives reflect smoothness)**.** *Assume the derivatives of j are uniformly injective near every point of U, and that j is C^2 and C^k. Let W be an open subset of a locally convex space and $\gamma\colon W \to U$ continuous. If $j \circ \gamma$ is of class C^k, then so is γ.*

Proof. By induction on k. Lemma 8.10, with $d_t = \gamma(q+th) - \gamma(q)$ and $\tau_t = t$, gives the derivative $\mathrm{d}\gamma$, which solves $\mathrm{d}j(\gamma q)(\mathrm{d}\gamma(q)(h)) = \mathrm{d}(j \circ \gamma)(q)(h)$; then Lemma 8.11 applies. □

Corollary 8.13 (Pointwise lower bounds in Banach spaces)**.** *Theorem 8.6 holds when $\mathrm{d}j(x)$ is bounded below at every $x \in U$, with a constant depending on x.*

Proof. Lemma 8.9 and Theorem 8.12; a Banach space is sequentially complete. □

Whether Theorem 8.12 holds when each $\mathrm{d}j(x)$ is only assumed to be a topological embedding is open.

Chapter 9

Analytic maps

We follow the treatment of Glöckner and Neeb [GN26, Ch. 2], which builds on Bochnak and Siciak [BS71a, BS71b] and on Glöckner [Glo02, §2]. The point of this approach is that *no completeness of the target space is needed* for the basic theory. Bochnak and Siciak assume throughout that the target space F is sequentially complete; Glöckner removes this hypothesis by working in the completion $\widetilde{F}$ and coming back to F through the derivatives, which lie in F [Glo02, Prop. 2.4, Lemma 2.5].

9.1 Continuous homogeneous polynomials

Definition 9.1 (Continuous homogeneous polynomials)**.** Let E, F be topological vector spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $k \in \mathbb{N}_0$. A map $p\colon E \to F$ is a *continuous homogeneous polynomial of degree* k if $p(x) = A(x, \dots, x)$ for some continuous k-linear map $A\colon E^k \to F$.

Replacing A by its symmetrization we may take A symmetric, and then A is determined by p:

Theorem 9.2 (Polarization; Mazur–Orlicz [BS71a, Theorem A])**.** *If A is symmetric and k-linear, then for $x_1, \dots, x_k \in E$*

$$\sum_{\varepsilon \in \{\pm 1\}^k} \varepsilon_1 \cdots \varepsilon_k \, A\Big(\sum_j \varepsilon_j x_j, \dots, \sum_j \varepsilon_j x_j\Big) = 2^k k! \, A(x_1, \dots, x_k).$$

In particular, if $x \mapsto A(x, \dots, x)$ is continuous, then A is continuous.

Proof. Expand $A(\sum_j \varepsilon_j x_j, \dots)$ by multilinearity over all maps $r\colon \{1, \dots, k\} \to \{1, \dots, k\}$. The sign sum $\sum_\varepsilon \prod_j \varepsilon_j^{1+|r^{-1}(j)|}$ is 2^k if r is a bijection and 0 otherwise, and $A(x_{r(1)}, \dots, x_{r(k)}) = A(x_1, \dots, x_k)$ for bijective r by symmetry. The continuity statement follows because the left hand side is a finite combination of values of $x \mapsto A(x, \dots, x)$. No completeness, local convexity or Hausdorff property is used. □

9.2 One complex variable

Let F be a complex locally convex space, not necessarily complete, and $U \subseteq \mathbb{C}$ open. A map $f\colon U \to F$ is *complex analytic* if around every point it is a power series $f(z_0 + z) = \sum_n z^n a_n$ with $a_n \in F$, converging pointwise on a disc [GN26, Rem. 2.1.3].

Theorem 9.3 (Power series in incomplete spaces)**.** *A complex analytic $f\colon U \to F$ is continuous, its complex derivative exists and is complex analytic. The Cauchy integrals $\int_0^{2\pi} e^{-int} f(z_0 + \rho e^{it})\,dt = 2\pi\rho^n a_n$ exist as weak integrals in F, the Cauchy estimates $q(a_n) \le \rho^{-n} \sup_{|\zeta|=\rho} q(f(z_0+\zeta))$ hold for every continuous seminorm q, and the Taylor series of f at z_0 converges to f in F on every disc whose closure lies in U ([GN26, Cor. 2.1.8, Lemma 2.1.10]).*

Proof. No completion is used. Every sum and integral that appears has its value already in F (a value of f, a coefficient, a derivative), and convergence is checked seminorm by seminorm against a geometric majorant. Scalar estimates are transferred to F by the Hahn–Banach theorem: if $|\ell(v)| \le B$ for every continuous linear ℓ with $|\ell| \le q$, then $q(v) \le B$. □

Theorem 9.4 (Complex analytic = real smooth with complex linear differential)**.** *$f\colon U \to F$ is complex analytic if and only if it is C^∞ in the real Michal–Bastiani sense and $\mathrm{d}f(z)(w) = w\,\mathrm{d}f(z)(1)$ for all $z \in U$, $w \in \mathbb{C}$ [GN26, Cor. 2.1.9].*

Remark 9.5. Glöckner and Neeb prove the one variable characterizations with C^1 in place of C^∞ under Mackey completeness [GN26, Prop. 2.1.6], and Corollary 2.1.9 without completeness through the completion $\widetilde{F}$. Here the smooth version is proved without completeness and without the completion: smoothness provides all the Taylor coefficients $\frac{1}{n!}\mathrm{d}^{(n)}f(z)(1,\dots,1)$ in F. For a single power series on a disc, the derivative at a point other than the center is only known to exist when the differentiated series converges in F; whether this can fail in an incomplete space we have not settled.

9.3 Complex analytic maps

Definition 9.6 (Complex analytic maps [GN26, Def. 2.1.2], [BS71b, Def. 5.6])**.** Let E, F be complex locally convex spaces and $U \subseteq E$ open. A map $f\colon U \to F$ is *complex analytic* if it is continuous and for every $x \in U$ there are continuous homogeneous polynomials p_k of degree k such that $f(x+y) = \sum_{k\ge 0} p_k(y)$, as a pointwise limit, for all y in a 0-neighborhood Y with $x + Y \subseteq U$.

Definition 9.7 (Gâteaux analytic maps [GN26, Def. 2.1.11])**.** f is *Gâteaux analytic* if $z \mapsto f(x+zy)$ is complex analytic (locally a power series with coefficients in F) on $\{z \in \mathbb{C} : x+zy \in U\}$, for all $x \in U$, $y \in E$.

Theorem 9.8 (Characterizations of complex analyticity [GN26, Thm. 2.1.12])**.** *For $f\colon U \to F$ the following are equivalent:*

1. *f is C^∞ (in the real Michal–Bastiani sense) and $\mathrm{d}f(x)\colon E \to F$ is complex linear for each $x \in U$;*
2. *f is continuous and Gâteaux analytic;*
3. *f is complex analytic;*
4. *f is $C^\infty_{\mathbb{C}}$, with complex difference quotients $t^{-1}(f(x+ty) - f(x))$, $t \in \mathbb{C}$.*

If F is Mackey complete, they are also equivalent to: f is $C^1_{\mathbb{C}}$; f is C^1 and $\mathrm{d}f(x)$ is complex linear for each $x \in U$.

Proof. The equivalence of (1)–(4) is formalized without any completeness of F and without the completion. (4) $\Rightarrow$ (1): complex difference quotients restrict to real ones, and substituting $t \mapsto ct$ gives complex homogeneity. (3) $\Rightarrow$ (4): the complex derivative of $z \mapsto f(x+zy)$ at 0 is $\mathrm{d}f(x,y)$, and $\mathrm{d}f$ is again complex analytic. For Mackey complete F, a C^1 map with complex linear differentials is weakly holomorphic along lines, hence Gâteaux analytic by the weak-to-strong theorem. For every $k \geq 1$, $C^k_{\mathbb{C}}$ maps are the C^k maps with complex linear differentials [GN26, Lemma 1.4.17]: writing $t = su$ with $s = |t|$ and $|u| = 1$, the real difference quotients converge to $\mathrm{d}f(x, uy)$ uniformly on the unit circle, and the second differential is complex linear in the direction by symmetry. (3) $\Rightarrow$ (2) restricts the expansion to lines. (1) $\Rightarrow$ (3): the Taylor polynomials $\frac{1}{n!}\mathrm{d}^{(n)}f(x)(y,\dots,y)$ are continuous homogeneous polynomials over $\mathbb{C}$ (complex multilinearity of the higher differentials from homogeneity and symmetry), and the one variable Taylor expansion along $z \mapsto f(x+zy)$ gives the series on a balanced Y with $x + 2Y \subseteq U$. (2) $\Rightarrow$ (1): $\mathrm{d}f$ is a locally uniform limit of difference quotients, by seminorm estimates of Taylor remainders along lines; it is Gâteaux analytic on $U \times E$ by a two variable Osgood lemma, and a discrete polarization over roots of unity writes its Taylor coefficients as finite combinations of coefficients of f along lines, so they lie in F. Induction on k gives smoothness. □

Theorem 9.9 (Holomorphic maps between complex normed spaces)**.** *Let E, F be complex normed spaces and $U \subseteq E$ open. A complex analytic map $f\colon U \to F$ is complex Fréchet differentiable; if F is complete, the converse holds. In particular, if F is complete, a locally uniform limit of complex differentiable maps on U is complex differentiable. If F is complete, complex differentiable maps on U are C^1 over $\mathbb{C}$.*

Proof. A complex analytic map is real Fréchet smooth (Theorem 9.8 and Proposition 2.3.17 of [GN26]), with complex linear derivative. Conversely, along each complex line a complex differentiable map is holomorphic in one variable, hence complex analytic along lines; being continuous, it is complex analytic by Theorem 9.8. For limits, the restrictions to complex lines are locally uniform limits of holomorphic functions of one variable, which are holomorphic, and the limit is continuous. □

Remark 9.10. A word of caution on the hypotheses. No completeness is needed for the equivalence of (1)–(3): in the proof of (2) $\Rightarrow$ (1), the Cauchy integral $\mathrm{d}f(x,y) = \frac{1}{2\pi i}\oint_{|\zeta|=1} f(x+\zeta y)\,\zeta^{-2}\,d\zeta$ is a weak integral which exists in F, because along the line it is a coefficient of a power series with coefficients in F. Completeness only enters when one starts from a C^1 map, whose higher derivatives are a priori only in the completion. Since (1) is stated with the real calculus, we do not need a separate complex Michal–Bastiani calculus: a complex C^k map is the same as a real C^k map with complex linear differentials.

9.4 Real analytic maps

Definition 9.11 (Complexification [GN26, Def. 2.2.1])**.** For a real topological vector space E, $E_{\mathbb{C}} := E \times E$ with the product topology and $(x+iy)(u,v) := (xu - yv, xv + yu)$. It is locally convex if E is, and $E = E \times \{0\}$.

Definition 9.12 (Real analytic maps; Milnor [Mil84], [Glo02, Def. 2.3], [GN26, Def. 2.2.2])**.** Let E, F be real locally convex spaces and $U \subseteq E$ open. A map $f\colon U \to F$ is *real analytic* if it extends to a complex analytic map $V \to F_{\mathbb{C}}$ on an open neighborhood V of U in $E_{\mathbb{C}}$.

Definition 9.13 (Analytic maps in the sense of Bochnak and Siciak [BS71b, Def. 5.6])**.** $f\colon U \to F$ is *BS analytic* if it is continuous and for every $x \in U$ there are continuous homogeneous

polynomials p_k of degree k such that $f(x+y) = \sum_{k\geq 0} p_k(y)$, as a pointwise limit, for all y in a 0-neighborhood. Over $\mathbb{C}$ this is Definition 9.6.

Compositions of complex analytic maps are complex analytic: by Theorem 9.8 this is the chain rule of the real Michal–Bastiani calculus, and no completeness is involved. In the same way complex analyticity is a local property.

Lemma 9.14 (Locality [GN26, Lemma 2.2.3])**.** *If every point of U has a neighborhood W such that $f|_{U\cap W}$ is real analytic, then f is real analytic.*

Proof. Each $x \in U$ has a convex open 0-neighborhood C_x with $x+C_x \subseteq U$ and a complex analytic extension g_x of f on the convex set $B_x := (x+C_x)+iC_x$. Two such extensions agree on $B_x \cap B_y$. Glöckner and Neeb deduce this from the identity theorem in $E_{\mathbb{C}}$ [GN26, Thm. 2.1.16(c)]. We use one variable instead: for $a+ib \in B_x \cap B_y$, the set D of $z \in \mathbb{C}$ with $a+zb \in B_x \cap B_y$ is convex and contains 0, i and a real interval around 0, on which $g_x(a+zb) = f(a+zb) = g_y(a+zb)$. For each continuous complex linear functional λ, the holomorphic functions $\lambda \circ g_x(a+zb)$ and $\lambda \circ g_y(a+zb)$ on D then agree, by the one variable identity theorem, in particular at $z = i$; and functionals separate points. The extensions glue to a complex analytic map on $\bigcup_x B_x$. □

Theorem 9.15 (Basic properties [GN26, Prop. 2.2.4, Lemmas 2.2.5, 2.2.6])**.** *Compositions of real analytic maps are real analytic. Real analytic maps are smooth, and $\mathrm{d}^{(k)} f(x, y_1, \dots, y_k) = \mathrm{d}^{(k)} g(x, y_1, \dots, y_k)$ for any complex analytic extension g, $x \in U$ and $y_i \in E$. A real analytic map is locally the sum of its Taylor series $\sum_k \frac{1}{k!}\mathrm{d}^{(k)} f(x)(y, \dots, y)$; in particular it is BS analytic.*

Proof. Composition: compose the extensions, after shrinking the domain of the inner one. Smoothness: $f = \mathrm{Re} \circ g \circ \iota$ on U, with $\iota\colon E \to E_{\mathbb{C}}$ and $\mathrm{Re}\colon F_{\mathbb{C}} \to F$ continuous and real linear (Glöckner and Neeb argue instead that F is closed in $F_{\mathbb{C}}$). The derivatives of $g \circ \iota$ at x are those of g in real directions, by an identity between difference quotients. The Taylor expansion is that of g (Theorem 9.8), restricted to real points. □

Proposition 9.16 (Operations on analytic maps)**.** *Pairs, sums, scalar multiples and finite sums of complex (resp. real) analytic maps are complex (resp. real) analytic, and so is $x \mapsto \beta(f(x), g(x))$ for a jointly continuous complex (resp. real) bilinear map β.*

Proof. In the complex case, by Theorem 9.8 it suffices to see that the maps are smooth with complex linear differentials, which follows from the rules of Section 3.3. In the real case, $(E \times F)_{\mathbb{C}} \cong E_{\mathbb{C}} \times F_{\mathbb{C}}$ as complex locally convex spaces, so the pair of two complex analytic extensions extends the pair; for β use its complexification $\beta_{\mathbb{C}}(a+ia', b+ib') = \beta(a,b) - \beta(a',b') + i(\beta(a,b') + \beta(a',b))$, which is jointly continuous and complex bilinear. □

Theorem 9.17 (The differential of a real analytic map)**.** *If $f\colon U \to F$ is real analytic on an open set $U \subseteq E$, then $\mathrm{d}f\colon U \times E \to F$ is real analytic. (Glöckner and Neeb use this, for $r = \omega$, in [GN26, Lemma 3.3.8(d)]; the complex case is [Glo02, Lemma 2.6].)*

Proof. Let g be a complex analytic extension of f on $V \subseteq E_{\mathbb{C}}$. Then $\mathrm{d}g$ is complex analytic on $V \times E_{\mathbb{C}}$ (Theorem 9.8), and composed with $(E \times E)_{\mathbb{C}} \cong E_{\mathbb{C}} \times E_{\mathbb{C}}$ it extends $\mathrm{d}f$, since $\mathrm{d}g(x,y) = \mathrm{d}f(x,y)$ for $x \in U$, $y \in E$ (Theorem 9.15). □

Proposition 9.18 (Complex analytic maps are real analytic [GN26, Prop. 2.2.11])**.** *Let E, F be complex locally convex spaces. Every complex analytic map $f\colon U \to F$ is real analytic, as a map between the underlying real spaces.*

Proof. Let $\pi_\pm(u+iv) := u \pm Jv$ on $E_{\mathbb{C}} = E \oplus iE$, where J is multiplication by i in E, and $\alpha(y) := \frac{1}{2}(y - iJy)$, $\beta(y) := \frac{1}{2}(y + iJy)$ from F to $F_{\mathbb{C}}$. Then π_+ and α are complex linear, π_- and β conjugate linear, and $g := \alpha \circ f \circ \pi_+ + \beta \circ f \circ \pi_-$ is smooth on $\pi_+^{-1}(U) \cap \pi_-^{-1}(U)$ with complex linear differentials, hence complex analytic by Theorem 9.8. Since $\pi_\pm(x) = x$ and $\alpha(y) + \beta(y) = y$ for real points, g extends f. This is the extension of Glöckner and Neeb, $\psi^{-1} \circ (f \times \bar{f}) \circ \varphi$, written without the conjugate spaces $\bar{E}$, $\bar{F}$. □

Lemma 9.19 (A complex norm on the complexification)**.** *Let E be a real normed space. Then $\|u+iv\|_{\mathbb{C}} := \sup_{|w|=1} \|\mathrm{Re}(w(u+iv))\|$ is a complex norm on $E_{\mathbb{C}}$ with $\max(\|u\|, \|v\|) \le \|u+iv\|_{\mathbb{C}} \le \|u\| + \|v\|$; it extends the norm of E, defines the product topology, and is complete if E is.*

Proof. The norm is induced by the injective map $p \mapsto (w \mapsto \mathrm{Re}(wp))$ into $C(S^1, E)$, and $\mathrm{Re}(wcp) = |c| \,\mathrm{Re}(w(c/|c|)p)$ gives $\|cp\|_{\mathbb{C}} \le |c| \|p\|_{\mathbb{C}}$. The bounds follow from $w = 1$, $w = -i$ and $|\mathrm{Re}\, w|, |\mathrm{Im}\, w| \le 1$. This is Taylor's complexification of the norm, the smallest reasonable complexification norm (Muñoz, Sarantopoulos and Tonge, Studia Math. 134 (1999), Proposition 3); Glöckner and Neeb use the largest complex norm extending $\|\cdot\|$ instead [GN26, Exercise 2.3.7]. □

Continuous linear maps and constants are real analytic.

Theorem 9.20 (Inversion in continuous inverse algebras [GN26, Cor. 1.3.8, Ex. 2.1.15, Lemmas 2.2.14, 2.2.17, Prop. 2.2.13])**.** *Let A be a continuous inverse algebra: a unital locally convex algebra whose unit group $A^\times$ is open, with inversion ι continuous on it (complete normed algebras are examples). Then ι is smooth with $\mathrm{d}\iota(x,h) = -x^{-1}hx^{-1}$; it is complex analytic if A is complex, and real analytic if A is real. No completeness is assumed.*

Proof. From $\iota(x+th) - \iota(x) = -t\,\iota(x+th)h\iota(x)$ we get the derivative, and $\mathrm{d}\iota(x,h) = -\iota(x)h\iota(x)$ is C^k when ι is (product rule), so ι is smooth. In the complex case $\mathrm{d}\iota(x,\cdot)$ is complex linear, hence ι is complex analytic by Theorem 9.8. For real A, the complexification $A_{\mathbb{C}}$ with $(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$ is a complex continuous inverse algebra: near 1, $(a+ib)^{-1} = (1+c^2)^{-1}(1-ic)a^{-1}$ with $c = a^{-1}b$ depends continuously on (a,b), and units forming a neighbourhood of 1, with inversion continuous at 1, suffice by translation. Inversion in $A_{\mathbb{C}}$ extends ι. □

Theorem 9.21 (Identity theorems [GN26, Thms. 2.1.16, 2.2.8])**.** *Let U be open and connected, and f, g complex (resp. real) analytic on U. Then $f = g$ if (a) they agree on a nonempty open subset, or (b) $\mathrm{d}^{(n)}f(x,y,\dots,y) = \mathrm{d}^{(n)}g(x,y,\dots,y)$ for all n, y and some $x \in U$, or, in the complex case, (c) they agree on $(x+H) \cap U$ for a real subspace H with $H + iH = E$. In particular, two complex analytic maps on a connected open subset of $E_{\mathbb{C}}$ which agree at its real points agree. No completeness is needed.*

Proof. The set of points near which $f = g$ is open; it contains x under (b), by the Taylor expansions (Theorems 9.8 and 9.15); it is relatively closed in U, since the differentials are continuous and vanish on it. Glöckner and Neeb treat the real case through a complex extension; the real Taylor expansion suffices. (a) implies (b). Under (c), for $y \in H$ the functions $z \mapsto \ell(f(x+zy))$ and $\ell(g(x+zy))$ agree on a real interval, hence near 0 by the one variable identity theorem, so the differentials agree on the diagonal of H^n; real polarization and complex multilinearity give (b). □

Proposition 9.22 (Complex extensions with values in a complex space [GN26, Prop. 2.2.12])**.** *A real analytic map $f\colon U \to F$ into a complex locally convex space extends to a complex analytic map $V \to F$ on an open set $V \subseteq E_{\mathbb{C}}$ containing U.*

Proof. Compose an extension $V \to F_{\mathbb{C}}$ with the complex linear map $F_{\mathbb{C}} \to F$, $v+iw \mapsto v+iw$. □

Theorem 9.23 (Analytic operator-valued maps)**.** *Let F_1, F_2 be normed spaces and $g\colon E \to \mathcal{L}(F_1, F_2)$, with $\mathcal{L}(F_1, F_2)$ carrying the operator norm.*

1. *If E is a complex locally convex space, F_1, F_2 are complex, $U \subseteq E$ is open and $(y, \xi) \mapsto g(y)\xi$ is complex analytic on $U \times F_1$, then $g\colon U \to \mathcal{L}_{\mathbb{C}}(F_1, F_2)$ is complex analytic.*
2. *If E is a real locally convex space, F_1, F_2 are real, $U \subseteq E$ is any subset and $(y, \xi) \mapsto g(y)\xi$ is real analytic on $U \times F_1$, then $g\colon U \to \mathcal{L}(F_1, F_2)$ is real analytic. Iterating, if $(y, \xi, \eta) \mapsto \Omega(y)\xi\eta$ is real analytic on $U \times F_1 \times F_2$, then $y \mapsto \Omega(y) \in \mathcal{L}(F_1, \mathcal{L}(F_2, F_3))$ is real analytic (e.g. the local representatives of a differential 2-form).*

No completeness is assumed.

Proof. Core step. Let Z be complex locally convex, F_1 real normed, F_2 complex normed and $K\colon Z \to \mathcal{L}_{\mathbb{R}}(F_1, F_2)$ such that $\hat{K}(z, \xi) = K(z)\xi$ is smooth on $U \times F_1$ with $\partial_{(ch,0)}\hat{K} = c\,\partial_{(h,0)}\hat{K}$ for $c \in \mathbb{C}$. By Theorem 8.4, K is smooth for the operator norm and $\mathrm{d}K(z, h)\xi = \partial_{(h,0)}\hat{K}(z, \xi)$, so $\mathrm{d}K(z, \cdot)$ is complex linear and K is complex analytic by Theorem 9.8.

(1) Apply the core step to $g(y)$ viewed as a real operator, and compose with the continuous complex linear projection $PT = \frac{1}{2}(T - iTi)$ of $\mathcal{L}_{\mathbb{R}}(F_1, F_2)$ onto $\mathcal{L}_{\mathbb{C}}(F_1, F_2)$, which fixes complex linear operators.

(2) Let H be a complex analytic extension of $\hat{g}$ on $V \subseteq (E \times F_1)_{\mathbb{C}}$. Since V need not contain $W \times F_1$ for an open $W \ni y$, we use the differential of H at $\xi = 0$: $K(z)\xi := \mathrm{d}H((z, 0), (0, \xi))$ for $z \in W := \{z \mid (z, 0) \in V\}$ and $\xi \in F_1$. Then $K(z) \in \mathcal{L}_{\mathbb{R}}(F_1, (F_2)_{\mathbb{C}})$, with the norm of Lemma 9.19, and the core step applies since $\mathrm{d}H$ is complex analytic (Theorem 9.17). For $y \in U$ the line $t \mapsto (y, t\xi)$ consists of real points, where H agrees with $\hat{g}$, so $K(y)\xi = g(y)\xi + i0$. The map $\mathcal{L}_{\mathbb{R}}(F_1, (F_2)_{\mathbb{C}}) \to \mathcal{L}(F_1, F_2)_{\mathbb{C}}$, $T \mapsto \mathrm{Re} \circ T + i\,\mathrm{Im} \circ T$, is continuous and complex linear, and carries K to a complex analytic extension of g. The bilinear statement follows by applying (2) to $(y, \xi) \mapsto \Omega(y)\xi$ and then to Ω. We do not claim this argument is new. □

Remark 9.24 (Two notions of real analyticity). Real analytic maps are BS analytic (Theorem 9.15). The converse holds when E is a Fréchet space [GN26, Rem. 2.2.7], and more generally when $E_{\mathbb{C}}$ is a Baire space [BS71b, Thm. 7.1], [Glo02, 2.9]; see Theorem 9.36. For BS analyticity, closedness under composition is only known when the spaces are Baire [BS71b, Thm. 7.3]; this is why the theory of analytic manifolds and Lie groups uses Milnor's definition.

9.5 Series of homogeneous polynomials

Where does the sum of a series $\sum_n p_n$ of continuous homogeneous polynomials, converging pointwise on an open set, define an analytic map? Bochnak and Siciak [BS71b, §5] assume F sequentially complete; the results below use no completeness of F.

Lemma 9.25 (Moving the centre of a bound)**.** *Let p be a continuous homogeneous polynomial of degree n between complex vector spaces, q a seminorm on F and $W \subseteq E$ balanced. If $q(p(c + w)) \le M$ for $w \in W$, then $q(p(w)) \le M$ for $w \in W$.*

Proof. With $\omega = e^{2\pi i/(n+1)}$, expanding the symmetric n-linear map gives $\sum_{j \le n} p(\omega^j c + w) = (n + 1)\,p(w)$, and $p(\omega^j c + w) = \omega^{jn} p(c + \omega^{-j} w)$ with $\omega^{-j} w \in W$. Over $\mathbb{R}$ or $\mathbb{C}$, [BS71a, Lemma 2 (ii)] gives the bound on $(2e)^{-1}W$. □

Proposition 9.26 (Normal convergence at zero [BS71b, Prop. 5.2])**.** *Let E be a complex Baire space and p_n continuous homogeneous polynomials of degree n such that $\sum_n p_n(a)$ converges for a in an absorbing set. For every continuous seminorm q on F and every $\theta > 0$ there are a 0-neighbourhood W and C with $q(p_n(w)) \le C\theta^n$ for all n and $w \in W$. If the series converges on a 0-neighbourhood, its sum is continuous at 0.*

Proof. The closed sets $\{x : q(p_n(x)) \le Nk^n \text{ for all } n\}$ cover E, so one of them has an interior point x_0; Lemma 9.25 moves the bound from $x_0 + B$ to the balanced core B, and homogeneity rescales it. Continuity at 0 follows from the geometric tail estimate and the continuity of the partial sums. □

Remark 9.27 (Counterexamples)*. E not Baire* [BS71b, Counterexample 5.1]: on the finitely supported sequences with the norm of ℓ^2, $\sum_n (nx_n)^n$ converges everywhere, but its sum is not continuous at 0. *F not complete, one variable*: in $F = \operatorname{span}\{e_n, (z^n)_n : |z| < 1\} \subseteq \ell^2$, the power series $\sum_n z^n e_n$ converges on the unit disc, but its sum has no complex derivative at $1/2$, since $\sum_n n2^{1-n}e_n \notin F$. A disc of convergence need not be a disc of analyticity. The partial sums of $\sum_n n2^{1-n}e_n$ are Mackey–Cauchy in F, so F is not Mackey complete (Definition 5.2).

Lemma 9.28 (Hartogs' lemma)**.** *Let $g_n \colon \mathbb{C} \to F$ be continuous with $\ell \circ g_n$ entire for every continuous functional ℓ, and q a continuous seminorm. If $q(g_n) \le e^{nB}$ on a closed disc $\bar{D}(z_0, R)$ and $(q(g_n(z)))_n$ is bounded at every point of the disc, then for every $\varepsilon > 0$ we have $q(g_n) \le e^{n\varepsilon}$ on $\bar{D}(z_0, R/2)$ for n large.*

Proof. The functions $v_n = n^{-1} \log^+ q(g_n)$ satisfy the sub-mean value inequality on circles: by Hahn–Banach $q(g_n(z)) = |\ell(g_n(z))|$ for a functional $\ell \le q$, and Jensen's formula gives $\log|\ell g_n(z)| \le$ the circle average of $\log|\ell \circ g_n|$. Integrating in polar coordinates, $\pi s^2 v_n(z) \le \int_{B(z,s)} v_n \le \int_{B(z_0,R)} v_n$ for $|z - z_0| \le s = R/2$. Since $0 \le v_n \le B$ and $v_n \to 0$ pointwise, dominated convergence makes the right-hand side small. □

Bochnak and Siciak use instead Siciak's polynomial lemma [BS71b, §4], which rests on potential theory (Siciak's extremal function, Fekete points); only lines are needed here.

Proposition 9.29 (Bounds along lines [BS71b, Prop. 4.1])**.** *If $\sum_n p_n$ converges pointwise on an open set H, then for every compact set $K \subseteq \mathbb{C}$ with $x + Ky \subseteq H$ there is $\theta < 1$ such that for every continuous seminorm q there is C with $q(p_n(x + zy)) \le C\theta^n$ for $z \in K$. The ratio θ does not depend on q.*

Proof. Near z_0, choose $\lambda > 1$ with $\lambda(x + zy) \in H$ on a disc. Then $g_n(z) = p_n(\lambda(x + zy))$ is pointwise bounded; Baire's theorem in $\mathbb{C}$ gives a small disc where it is uniformly bounded, and the Cauchy estimates for polynomials (by discrete Fourier inversion over roots of unity) give the Bernstein–Walsh bound $q(g_n) \le e^{nB}$ on the whole disc. Hartogs' lemma with $\varepsilon < \log\lambda$ gives $q(p_n(x + zy)) = \lambda^{-n} q(g_n(z)) \le (e^{\varepsilon}/\lambda)^n$ for large n. Only the constant and the threshold depend on q: λ depends on H along the line and $\varepsilon = \frac{1}{2}\log\lambda$. Cover K by finitely many discs. □

Theorem 9.30 (Pointwise limits of polynomials [BS71a, Thm. 2])**.** *Over a Baire space, a pointwise limit of continuous homogeneous polynomials of degree k is a continuous homogeneous polynomial of degree k. No completeness of F is needed.*

Proof. The multilinear map is the limit of the polarizations. By Baire and Lemma 9.25 the limit P is bounded on a balanced 0-neighbourhood; the Cauchy estimates for $t \mapsto P(x_0 + th)$ then give continuity at every x_0. □

Theorem 9.31 (Bochnak–Siciak [BS71b, Thms. 5.1, 5.2])**.** *Let E be a complex locally convex Baire space and F a Mackey complete complex locally convex space (for instance a sequentially complete one). If a series $\sum_n p_n$ of continuous homogeneous polynomials converges on an open set H, then its sum is complex analytic on H.*

Proof. By Proposition 9.29 the series converges locally uniformly with geometric rate on every complex line, so the sum is Gâteaux analytic, with Taylor coefficients $T_k(h) = \sum_n (\text{coefficient of } t^k \text{ in } p_n(x+th))$. These series have geometric bounds $C_q\theta^n$ with θ independent of q, so their partial sums are Mackey–Cauchy and the $T_k(h)$ lie in F. By Theorem 9.30 each T_k is a continuous homogeneous polynomial, and $f(x+h) = \sum_k T_k(h)$ near $h = 0$, so f is continuous at x by Proposition 9.26. Theorem 9.8 concludes. □

Both hypotheses are needed (Remark 9.27). Bochnak and Siciak assume F sequentially complete; Mackey completeness is strictly weaker.

9.6 Series of real homogeneous polynomials

Let E, F be real locally convex spaces. A continuous real n-linear map A extends to a continuous complex n-linear map $A_{\mathbb{C}}(z_1, \dots, z_n) = \sum_{S \subseteq \{1,\dots,n\}} i^{|S|} A(w_S)$, where $(w_S)_j = \operatorname{Im} z_j$ for $j \in S$ and $\operatorname{Re} z_j$ otherwise; the diagonal of $A_{\mathbb{C}}$ is the complexification $p_{\mathbb{C}}$ of the homogeneous polynomial p of A.

Lemma 9.32 (Complexification of polynomials)**.** *$A_{\mathbb{C}}$ is complex multilinear and continuous, and extends A. Hence $p_{\mathbb{C}}$ is a continuous complex homogeneous polynomial of degree n on $E_{\mathbb{C}}$ with $p_{\mathbb{C}}(x) = p(x)$ for $x \in E$.*

Proof. Real multilinearity and continuity are clear. In the j-th slot, $A_{\mathbb{C}}(\dots, u, \dots) = R(\operatorname{Re} u) + iR(\operatorname{Im} u)$ with R real linear (pair $S \not\ni j$ with $S \cup \{j\}$), which commutes with multiplication by i. □

Lemma 9.33 (Forward differences)**.** *For a homogeneous polynomial p of degree n and any c, v, $\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} p(c + jv) = n!\, p(v)$. Consequently, if $(q(p_n(sx + ty)))_n$ is bounded at each point of the square $|s|, |t| < 2$, then $q(p_n(sx + ty)) \le MK^n$ for $|s|, |t| \le 1$, with M, K depending on q.*

Proof. The identity is the n-th forward difference of $t \mapsto p(c + tv)$. Baire's theorem in $\mathbb{R}^2$ gives a disc $B(c, \rho)$ in the (s, t)-plane where $q \circ p_n \le m$ for all n; the identity bounds $q(p_n(v)) \le 2^n m/n!$ for $|v| < \rho/(n+1)$, and homogeneity and $(n+1)^n/n! \le e^{n+1}$ give the claim. □

The key observation replaces Siciak's polynomial lemma. For $x, y \in E$, the complex line $\lambda \mapsto (x + iy) + \frac{\lambda}{2}(x - iy)$ meets the complex multiples of real vectors on the circle $|\lambda| = 2$:

$$(x + iy) + e^{i\theta}(x - iy) = 2e^{i\theta/2}\big(\cos(\theta/2)x + \sin(\theta/2)y\big).$$

On this circle $p_{\mathbb{C}}$ is therefore determined by the values of p at real points: $\|p_{\mathbb{C}}(\cdots)\| = 2^n \|p(\cos(\theta/2)x + \sin(\theta/2)y)\|$.

Lemma 9.34 (Jensen's inequality on the auxiliary line)**.** *Let p be a continuous real homogeneous polynomial of degree n, q a continuous seminorm on F and $\kappa > 0$. If v is the real or the imaginary part of $p_{\mathbb{C}}(x + iy)$, then*

$$\log^+ \frac{q(v)}{\kappa^n} \le \frac{1}{2\pi} \int_0^{2\pi} \log^+ \Big(\big(\tfrac{2}{\kappa}\big)^n q(p(\cos(\theta/2)x + \sin(\theta/2)y)) \Big)\, d\theta.$$

Proof. Take a real functional φ with $\varphi(v) = q(v)$ and $|\varphi| \le q$ (Hahn–Banach) and its complexification $\varphi_{\mathbb{C}}$, so that $q(v) \le |\varphi_{\mathbb{C}}(p_{\mathbb{C}}(x+iy))|$. Apply the sub-mean value inequality for $\log^+$ of the polynomial $\lambda \mapsto \kappa^{-n}\varphi_{\mathbb{C}}(p_{\mathbb{C}}((x+iy)+\frac{\lambda}{2}(x-iy)))$ on the circle $|\lambda| = 2$, and the identity above. □

Proposition 9.35 (Bochnak–Siciak [BS71b, Prop. 5.4])**.** *Let $\sum_n p_n$ be a series of continuous real homogeneous polynomials such that $(q(p_n(y)))_n$ is bounded for every continuous seminorm q and every y in a 0-neighborhood Y. Then there is a 0-neighborhood Z in $E_{\mathbb{C}}$ such that $q(\operatorname{Re} p_{n,\mathbb{C}}(z)), q(\operatorname{Im} p_{n,\mathbb{C}}(z)) \le C(9/10)^n$ for $z \in Z$, with C depending on z and q only. If F is Mackey complete, the complexified series converges on Z. No Baire assumption on E is needed.*

Proof. Choose a continuous seminorm N and $r > 0$ with $\{N < r\} \subseteq Y$, and let Z be the set of $x + iy$ with $N(x), N(y) < r/5$. Then $sx + ty \in Y$ for $|s|, |t| < 2$, and $\frac{5}{2}(\cos(\theta/2)x + \sin(\theta/2)y) \in Y$. Apply Lemma 9.34 with $\kappa = 4/5$: the functions $f_n(\theta) = \frac{1}{n}\log^+((5/2)^n q(p_n(\cos(\theta/2)x + \sin(\theta/2)y)))$ tend to 0 pointwise, and are bounded uniformly by Lemma 9.33. By dominated convergence their averages tend to 0, so $q(v_n) \le (4/5)^n(9/8)^n = (9/10)^n$ for large n. The ratio does not depend on q, so the partial sums are Mackey–Cauchy in $F_{\mathbb{C}}$, which is Mackey complete with F. □

Bochnak and Siciak prove Proposition 9.35 with Siciak's polynomial lemma for a real segment [BS71b, §4]: a family of polynomials which is pointwise bounded on $[-1, 1]$ is bounded by $M\omega^n$ on a neighborhood of $[-1, 1]$ depending only on $\omega > 1$. Its proof needs potential theory (extremal functions of increasing unions of compact sets). The auxiliary line avoids it: the homogeneity of p_n in two real variables brings the complex points into play.

Theorem 9.36 (Bochnak–Siciak [BS71b, Thm. 7.1])**.** *Let E be a real locally convex space such that $E_{\mathbb{C}} = E \times E$ is a Baire space (e.g. a Fréchet space) and F a Mackey complete real locally convex space. Then a map $f\colon U \to F$ is real analytic if and only if it is BS analytic.*

Proof. Around $a \in U$, the BS series converges on a 0-neighborhood, so its terms are bounded there. By Proposition 9.35 the complexified series converges on a 0-neighborhood Z of $E_{\mathbb{C}}$, and its sum is complex analytic on the interior of Z by Theorem 9.31. Translated to a, it extends f near a; Lemma 9.14 glues the local extensions. The converse is Theorem 9.15. □

9.7 Separately holomorphic maps

Hartogs' theorem asserts that a function of two complex variables which is holomorphic in each variable separately is continuous, hence holomorphic; no continuity is assumed. We prove the continuity for maps with values in a complex locally convex space F, without completeness, following Mujica [Muj86, §36] one continuous seminorm q at a time: continuity in F is continuity for every q.

Lemma 9.37 (Baire's theorem for separately continuous maps [Muj86, Lemma 36.2])**.** *If $g(\zeta, \sigma)$ is continuous in $\zeta \in B(a, R)$ and in $\sigma \in \bar{B}(c, S)$ separately, then $q \circ g$ is bounded on $B(b, \rho) \times \bar{B}(c, S)$ for some ball $B(b, \rho) \subseteq B(a, R)$.*

Proof. The closed sets $\{\zeta : q(g(\zeta, \sigma)) \le m \text{ for } \sigma \in \bar{B}(c, S)\}$ cover $B(a, R)$. □

Lemma 9.38 (Weakly holomorphic maps and seminorms)**.** *Let $A\colon B(c, 3r) \to F$ be such that $\ell \circ A$ is holomorphic for every continuous functional ℓ with $|\ell| \le q$, and $q \circ A \le M$. Then A is q-Lipschitz on $\bar{B}(c, r)$ with constant M/r, and $\log^+ q(A)$ has the sub-mean value property. Hartogs' lemma (Lemma 9.28) holds for nonnegative continuous functions on a disc with the sub-mean value property.*

Proof. Hahn–Banach reduces both statements to the functionals ℓ dominated by q: the Cauchy estimate for $(\ell \circ A)'$, and Jensen's inequality for $\ell \circ A$ with ℓ attaining q at the centre. For Hartogs' lemma, extend the functions continuously to $\mathbb{C}$ (Tietze) and argue as before. □

Theorem 9.39 (Hartogs [Muj86, Lemma 36.3, Thm. 36.1])**.** *Let $U \subseteq \mathbb{C}^2$ be open and $g\colon U \to F$ complex analytic in each variable separately. Then g is continuous. No completeness of F is needed.*

Proof. Fix q and a bidisc of radius $4R$ about (a,c) in U. By Lemma 9.37, $q(g) \le m$ on $B(b,\rho) \times \bar{B}(c,3R)$ with $|b-a| < R$. Let $A_k(\sigma)$ be the Taylor coefficients of $g(\cdot,\sigma)$ at b. The Cauchy estimates on a small circle give $q(A_k) \le m/r_0^k$; on the circle of radius $2R$, $(2R)^k q(A_k(\sigma))$ is bounded for each σ. For ℓ dominated by q, $\ell \circ A_k$ is holomorphic by Osgood's lemma, so Lemma 9.38 applies to $v_k = k^{-1}\log^+((2R)^k q(A_k))$: eventually $q(A_k(\sigma)) \le (2/(3R))^k$ for $|\sigma - c| \le R/2$. Hence the Taylor series $\sum_k (\zeta-b)^k A_k(\sigma)$ has terms bounded by $(5/6)^k$ near (a,c), and each term is q-continuous, so g is q-continuous at (a,c). □

Proposition 9.40 (Weak holomorphy in Mackey complete spaces)**.** *Let F be Mackey complete and $f\colon U \to F$ continuous on an open set $U \subseteq \mathbb{C}$, with $\ell \circ f$ holomorphic for every continuous functional ℓ. Then f is complex analytic.*

Proof. With $\omega = e^{2\pi i/N}$ and $j < N$, the average $S_N = N^{-1}\sum_{i<N} \omega^{i(N-j)} f(x + \frac{\rho}{2}\omega^i) \in F$ satisfies $|\ell(S_N) - (\rho/2)^j c_j^\ell| \le 2M_\ell 2^{-N}$, where c_j^ℓ is the j-th Taylor coefficient of $\ell \circ f$ at x and M_ℓ bounds $|\ell \circ f|$ on the circle of radius ρ: only the coefficients of index $\ge N$ alias onto c_j. By Hahn–Banach, $q(S_{N+1} - S_N) \le 4M_q 2^{-N}$ for every continuous seminorm q, with the ratio $1/2$ independent of q, so (S_N) converges in F. Its limit gives the coefficient $a_j \in F$ with $\ell(a_j) = c_j^\ell$, and the Taylor series converges in F (`WeakTaylorOn.hasPowerSeriesOnDisc`). □

Theorem 9.41 (Hartogs, joint analyticity [Muj86, Thm. 36.1])**.** *If F is Mackey complete, a map $g\colon U \to F$ on an open set $U \subseteq \mathbb{C}^2$ which is complex analytic in each variable separately is complex analytic.*

Proof. By Theorem 9.39, g is continuous. For each functional ℓ, $\ell \circ g$ is continuous and separately holomorphic, so by Osgood's lemma it has a double power series and is holomorphic along every complex line (swap the variables if the slope exceeds 1). By Proposition 9.40, g is complex analytic along every complex line, i.e. Gâteaux analytic, and Theorem 9.8 concludes. □

Example 9.42 (Separate does not imply joint analyticity without completeness). Let $F \subseteq \mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ (product topology) be the span of the matrices $\alpha(n)\sigma^m$ and $\zeta^n\beta(m)$, and $G(\zeta,\sigma) = (\zeta^n\sigma^m)$. Then $G\colon \mathbb{C}^2 \to F$ is complex analytic in each variable (and continuous), but not complex analytic. In particular F is not Mackey complete.

Proof. The Taylor coefficients of G in ζ at any point, $\binom{n}{k}\zeta_0^{n-k}\sigma^m$, and those in σ, lie in F, and the series converge coordinatewise. The second Taylor coefficient of $t \mapsto G(t,t)$ is $e_{20}+e_{11}+e_{02}$. If it were a combination of matrices $\alpha_i(n)\sigma_i^m$ and $\zeta_j^n\beta_j(m)$, take $P = X\prod_{\sigma_i\neq 0}(X-\sigma_i)$ and $Q = X\prod_{\zeta_j\neq 0}(X-\zeta_j)$. The form $x \mapsto \sum_{s,r} q_s p_r x(s,r)$ vanishes on $\alpha(n)\sigma_i^m$ (it is $P(\sigma_i)\sum_s q_s\alpha(s)$) and on $\zeta_j^n\beta(m)$, but equals $q_1p_1 \neq 0$ on $e_{20}+e_{11}+e_{02}$, since $p_0 = q_0 = 0$. The last claim follows from Theorem 9.41. □

So Mackey completeness enters only through the Taylor coefficients along oblique lines, and some completeness is needed there; continuity needs none.

Theorem 9.43 (Products [Muj86, Thms. 36.5, 36.8])**.** *A Gâteaux analytic map which is locally bounded for every continuous seminorm is continuous, hence complex analytic; no completeness is needed. If F is Mackey complete, a separately complex analytic map on an open subset of $E_1 \times E_2$ is Gâteaux analytic; if it is also locally bounded, it is complex analytic.*

Proof. If $q(f) \le M$ on $a + W$, take a balanced V with $a + \lambda h \in W$ for $h \in V$, $|\lambda| \le 3$; along $\lambda \mapsto f(a+\lambda h)$ the Cauchy estimates give $q(f(a+th) - f(a)) \le M|t|$ for $|t| \le 1$ (Lemma 9.38), so f is q-continuous at a. For the products, $(\zeta, \sigma) \mapsto f(p_1 + \zeta u, p_2 + \sigma v)$ is separately analytic on an open subset of $\mathbb{C}^2$, hence complex analytic by Theorem 9.41, and $f(p + z(u, v))$ is its restriction to the diagonal. □

Theorem 9.44 (Zorn's theorem for one seminorm [Muj86, Lemma 36.6, Prop. 36.7, Thm. 36.5])**.** *Let E be a Baire space, f Gâteaux analytic on U, q a continuous seminorm on F, and let the N-ball of radius 8ρ about c lie in U. If $q \circ f$ is bounded near $c - e$ with $N(e) < \rho/16$, then it is bounded near c. No completeness of F is needed.*

Proof. Let $T_y(k, h)$ be the coefficient of λ^k in $f(y + \lambda h)$. On each complex plane, f is continuous (Theorem 9.39), so by Osgood's lemma $\lambda \mapsto \ell(T_y(m, h+\lambda v))$ is a polynomial of degree $\le m$ whose coefficients are the double Taylor coefficients c_{ab} of $(\zeta, \sigma) \mapsto \ell f(y + \zeta h + \sigma v)$, and $\ell(T_{y+h}(k, v)) = \sum_b c_{kb}$. Hence: if $T_y(m, \cdot)$ is q-bounded on a ball, it is q-continuous (its coefficients along planes are bounded); and a bound on $t_0 + V$ passes to V by averaging over roots of unity (shift lemma). With $x = c - e$: the $T_x(m, \cdot)$ are q-bounded on a ball by the Cauchy estimates, hence q-continuous. The maps $Q_m(t) = (m+1)^{-1} \sum_j \omega^{j(m+1-k)} T_x(m, e + \omega^j t)$, $\omega = e^{2\pi i/(m+1)}$, satisfy $\ell(Q_m(t)) = c_{k,m-k}$, so their partial sums are q-continuous, pointwise bounded, and converge weakly to $T_c(k, t)$. Baire's theorem and the shift lemma bound each $T_c(k, \cdot)$ on a ball, so they are q-continuous. A second application of Baire's theorem to $(2^k T_c(k, \cdot))_k$ and the shift lemma give $q(T_c(k, v)) \le n 2^{-k}$ near 0, hence $q(f(c + v)) \le 2n$. □

Theorem 9.45 (Hartogs' theorem on products [Muj86, Lemma 36.2, Thm. 36.8])**.** *Let E_1 and $E_1 \times E_2$ be Baire spaces, E_2 first countable (e.g. Fréchet spaces), and F Mackey complete. A separately complex analytic map on an open subset of $E_1 \times E_2$ is complex analytic.*

Proof. Fix (a, b) and q. With a countable decreasing basis (V_n) at b, the closed sets $\{x : q(f(x, y)) \le n \text{ for } y \in V_n\}$ cover a neighbourhood of a, so by Baire's theorem $q \circ f$ is bounded near some (x', b) with x' as close to a as needed. Theorem 9.44 gives a bound near (a, b), and Theorem 9.43 concludes. □

9.8 Comparison with Mathlib on Banach spaces

On Banach spaces, Mathlib's `AnalyticOnNhd` (convergent power series in norm) should be compared with the definitions above, as Theorem 8.1 compares the Michal–Bastiani and Fréchet calculi. This is planned once the locally convex theory is in place.

Chapter 10

Differential equations

Differential equations are where the Banach and the locally convex settings part ways. On a Banach space the classical theory holds: local solutions exist, are unique, and depend smoothly (or analytically) on time and initial value. On a Fréchet space even linear equations with continuous coefficients may have no solution, or more than one (Chapter 16). What survives is the case in which the state space is a Banach space and only the *parameters* live in a locally convex space; this is the case treated in [GN26, §§2.4–2.5].

We first treat the second order equation of a quadratic spray, which is what the geodesics of lc-manifolds need, then local flows of vector fields on Banach spaces with C^n, holomorphic and real analytic dependence, and finally equations with locally convex parameters and their maximal flows.

10.1 Quadratic second order equations

On a Banach space E we consider the equation of the geodesics of a spray,

$$c'' = \Gamma(c)(c', c'),$$

where $\Gamma\colon U \times E \times E \to E$ is C^2 in the sense of Michal–Bastiani on an open set $U \subseteq E$ and bilinear in the last two variables. It is equivalent to the first order system $(c, v)' = (v, \Gamma(c)(v, v))$, whose right hand side is Fréchet C^1 (Theorem 8.1), hence locally Lipschitz.

Theorem 10.1 (Local existence, uniform for bounded velocities)**.** *Let E be a Banach space and $x_0 \in U$. There is $r > 0$ such that for every $K \geq 0$ there is $\varepsilon > 0$ for which every initial condition $c(0) = y$, $c'(0) = w$ with $\|y - x_0\| \leq r$ and $\|w\| \leq K$ has a solution on $[-\varepsilon, \varepsilon]$ staying in U.*

Proof. The Picard–Lindelöf theorem, keeping track of the fact that the solution stays in the ball where the vector field is Lipschitz, gives a uniform time for $\|w\| \leq \delta$. If c is a solution, so is $t \mapsto c(st)$, with initial velocity $s\,c'(0)$; rescaling by $s = (K + \delta)/\delta$ gives the statement. □

Theorem 10.2 (Uniqueness)**.** *Two solutions on (a, b), staying in U, with the same position and velocity at some $t_0 \in (a, b)$, agree on (a, b).*

Proof. Integral curves of a locally Lipschitz autonomous field agree near every point where they meet (Grönwall); the set where they agree near t is open and closed in (a, b). □

10.2 Local flows on Banach spaces

Let E be a Banach space, $U \subseteq E$ open and $f\colon U \to E$ Fréchet C^n, $1 \le n \le \infty$. We prove that the solutions of $c' = f \circ c$ depend in a C^n way on time and initial value, following the implicit function approach of [Lan99, Ch. IV, §1].

Lemma 10.3 (Superposition operators)**.** *Let K be compact. If f is C^n on U, then $\gamma \mapsto f \circ \gamma$ is Fréchet C^n on the open set of curves $\gamma \in C(K, E)$ with values in U (sup norm), with derivative $\eta \mapsto (s \mapsto f'(\gamma(s))\,\eta(s))$.*

Proof. The derivative comes from the mean value inequality and the uniform continuity of f' near the compact set $\gamma(K)$ (compare [Lan99, Lemma IV.1.12]). It is again a superposition operator, composed with a continuous bilinear map, so induction on n applies. □

Theorem 10.4 (Local flows with C^n dependence)**.** *For $x_0 \in U$ there are $\varepsilon > 0$, an open $V \ni x_0$ and $\mathrm{Fl}\colon (-\varepsilon, \varepsilon) \times V \to U$, jointly C^n, with $\mathrm{Fl}(0, x) = x$ and $\partial_t \mathrm{Fl}(t, x) = f(\mathrm{Fl}(t, x))$. If f is smooth in the sense of Michal–Bastiani, so is* Fl.

Proof. Let $J\gamma(s) = \int_0^s \gamma$ on $C([-1, 1], E)$ and $\Psi(\lambda, x, \gamma) = (\lambda, x, \gamma - x - \lambda J(f \circ \gamma))$. By Lemma 10.3, Ψ is C^n, and its derivative at $(0, x_0, x_0)$ is a shear, hence invertible. The inverse function theorem gives a C^n family $\Gamma(\lambda, x)$ with $\Gamma = x + \lambda J(f \circ \Gamma)$; put $\mathrm{Fl}(t, x) = \Gamma(t, x)(1)$. By injectivity of Ψ near $(0, x_0, x_0)$, $\Gamma(\sigma\lambda, x)(1) = \Gamma(\lambda, x)(\sigma)$ for $|\sigma| \le 1$, and the fundamental theorem of calculus gives the equation. Lang applies the implicit mapping theorem only in x, on a fixed short time interval [Lan99, Theorem IV.1.14]; taking the time scale λ as a parameter gives joint smoothness directly. We do not claim this argument is new. □

Theorem 10.5 (Complex time)**.** *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, E a Banach space over $\mathbb{K}$ and f of class C^n over $\mathbb{K}$ on U, $1 \le n \le \infty$. For $x_0 \in U$ there are $\varepsilon > 0$, an open $V \ni x_0$ and $\mathrm{Fl}\colon B_{\mathbb{K}}(0, \varepsilon) \times V \to U$, jointly C^n over $\mathbb{K}$, with $\mathrm{Fl}(0, x) = x$, whose restriction to real times solves $\partial_t \mathrm{Fl}(t, x) = f(\mathrm{Fl}(t, x))$.*

Proof. The proof of Theorem 10.4 works verbatim over $\mathbb{K}$: the curves are parametrized by the real interval $[-1, 1]$ and only the time scale λ lies in $\mathbb{K}$; Mathlib's inverse function theorem holds over $\mathbb{R}$ and $\mathbb{C}$. The rescaling identity is used for real σ and real times. Theorem 10.4 is the case $\mathbb{K} = \mathbb{R}$. □

Theorem 10.6 (Holomorphic and real analytic flows [GN26, Theorem 2.4.34])**.** *If f is complex differentiable on an open subset U of a complex Banach space, its local flows are complex differentiable in complex time and initial value. If f is real analytic on an open subset U of a real Banach space, its local flows $\mathrm{Fl}\colon (-\varepsilon, \varepsilon) \times V \to U$ are real analytic.*

Proof. The holomorphic case is Theorem 10.5 for $\mathbb{K} = \mathbb{C}$, since complex differentiable maps are C^1 over $\mathbb{C}$ (Theorem 9.9). In the real analytic case, let g be a complex analytic extension of f to an open subset of $E_{\mathbb{C}}$, a complex Banach space with the norm of Lemma 9.19, and Φ its holomorphic flow. For real t and $x \in E$, $t \mapsto \mathrm{Fl}(t, x)$ and $t \mapsto \Phi(t, x)$ solve $c' = g \circ c$ with $c(0) = x$, since g extends f; by uniqueness they agree. So Φ, composed with $(\mathbb{R} \times E)_{\mathbb{C}} \cong \mathbb{C} \times E_{\mathbb{C}}$, is a complex analytic extension of Fl. This is the proof of Glöckner and Neeb (real analytic case, p. 206), except that they show that complex solutions with real data stay real (their Lemma 2.4.30), while we use the real flow and uniqueness. □

Corollary 10.7 (Time-dependent equations with parameters [GN26, Theorem 2.4.34(b)])**.** *Let E, Λ be real Banach spaces and f real analytic on an open subset U of $\mathbb{R} \times E \times \Lambda$. Near every $p_0 \in U$ the solution of $c'(t) = f(t, c(t), \lambda)$ with $c(t_0) = x$ is real analytic in (t, t_0, x, λ).*

Proof. Apply Theorem 10.6 to the real analytic field $(s, x, \lambda) \mapsto (1, f(s, x, \lambda), 0)$. Along its integral curves the first component has derivative 1 and the last one derivative 0, so the curve through (t_0, x, λ) is $t \mapsto (t_0{+}t, c(t_0{+}t), \lambda)$. Glöckner and Neeb allow any locally convex parameter space; that case is Theorem 10.11. □

Theorem 10.8 (Uniqueness)**.** *If f is C^1 on U, two solutions of $c' = f \circ c$ on (a, b) with values in U that agree at one instant agree on (a, b).*

Proof. f is locally Lipschitz; apply the argument of Theorem 10.2. □

10.3 Parameters in locally convex spaces and maximal flows

Outside Banach spaces local existence and uniqueness can fail, even for continuous linear vector fields [GN26, Examples 2.4.2 and 2.4.3].

Lemma 10.9 (Uniform contractions with parameters [GN26, Lemma 2.3.26(a), Proposition 2.3.27(a)])**.** *Let $f(p, x)$ be a family of θ-contractions ($\theta < 1$) of a closed set of a Banach space F, with p in an open subset of a topological vector space. The fixed point $\varphi(p)$ is continuous, and if f is C^1 so is φ, with $\mathrm{d}\varphi(p, q) = \sum_n A_p^n b_p(q)$, $A_p = \mathrm{d}_2 f(p, \varphi(p))$, $b_p = \mathrm{d}_1 f(p, \varphi(p))$. If f is C^k, so is φ.*

Proof. Continuity from the uniform contraction estimate; the derivative is the Neumann series, whose terms converge locally uniformly (Tannery's theorem). For C^k, by induction on k: differentiating $\varphi(p) = f(p, \varphi(p))$, $\mathrm{d}\varphi(p, q)$ is the fixed point of the uniform family of θ-contractions $w \mapsto \mathrm{d}f((p, \varphi(p)), (q, w))$ with parameter (p, q), which is C^{k-1} by the induction hypothesis applied to φ. □

Lemma 10.10 (C^k superposition operators with parameters [GN26, Proposition 1.7.5])**.** *Let K be a compact metric space, $\kappa \colon K \to \mathbb{R}$ continuous, and H a C^k map on an open subset of $\mathbb{R} \times (E' \times Z)$ containing $\kappa(K) \times \Omega$, E' normed. The operator $(\psi, z) \mapsto (s \mapsto H(\kappa(s), \psi(s), z))$ is C^k on the open set of $(\psi, z) \in C(K, E') \times Z$ with $(\psi(s), z) \in \Omega$ for all s.*

Proof. The difference quotients converge uniformly in s by the mean value inequality. By induction on k: the derivative is the superposition operator of $\mathrm{d}H(\cdot, (0, \cdot))$, read on $C(K, E' \times E') \times (Z \times Z)$. □

Theorem 10.11 (Solutions with locally convex parameters [GN26, Theorem 2.4.34])**.** *Let E be a complex Banach space, W a complex locally convex space and f complex analytic on an open subset of $\mathbb{C} \times E \times W$. The solution $\Phi(t, t_0, y_0, p)$ of $y' = f(t, y, p)$, $y(t_0) = y_0$, is complex analytic in all variables near any point. The same holds for real analytic f on $\mathbb{R} \times E \times W$, E a real Banach space and W a real locally convex space, and if f is C^k, $k \geq 1$, the solution is C^k in all variables; if f is continuous and Lipschitz in y, uniformly in the other variables near the point, the solution is continuous.*

Proof. As in [GN26]: $\eta(\tau) = \Phi(t_0{+}\tau(t{-}t_0), \ldots){-}y_0$ is the fixed point of $\eta \mapsto \int_0^\tau (t{-}t_0) f(t_0{+}s(t{-}t_0), \eta(s) + y_0, p)\, ds$ on $C([-1, 1], E)$, a uniform family of contractions, C^1 by the superposition results; its differential is complex linear, and a real C^1 map with complex linear differentials into a Banach space is complex analytic [GN26, Theorem 2.1.12]. The rescaling identity (2.107) gives the differential equation. In the real case, the complexified equation preserves the closed set of curves with values in the real subspace, so its fixed point is real for real data; this replaces [GN26, Lemma 2.4.30]. In the C^k case the map g is C^k by Lemma 10.10, hence so is its fixed point (Lemma 10.9); the construction is written once for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. For continuous f, the bounds come from continuity and the Lipschitz condition, and $\Phi(r) = y_0 + \int_{t_0}^{r} f(v, \Phi(v), p)\, dv$. □

Theorem 10.12 (Maximal flows [GN26, Theorem 2.4.29, Lemma 2.5.7, Theorem 2.5.18, Remark 2.4.35])**.** *Let E be a real Banach space, W a topological space and f defined on $U \subseteq \mathbb{R} \times E \times W$, locally Lipschitz in y. Solutions of $y' = f(t, y, p)$ on open intervals are unique and glue to maximal solutions; let $\mathrm{Fl}(t, t_0, y_0, p)$ be the maximal flow, on its domain Ω. Then $\mathrm{Fl}(t, t_2, \mathrm{Fl}(t_2, t_1, y_1, q), q) = \mathrm{Fl}(t, t_1, y_1, q)$. If the equation admits local flows of a regularity class (closed under restriction, gluing and the composition below), Ω is open and* Fl *has that regularity on it. In particular, if W is a real locally convex space and f is continuous and locally Lipschitz in y (with parameters), C^k ($k \geq 1$), smooth or real analytic on an open set U, then Ω is open and* Fl *has the same regularity on Ω; if all solutions with initial data in $J \times S$ exist on the open interval J,* Fl *is real analytic on $J \times J \times S$.*

Proof. Uniqueness on open intervals by connectedness, from local uniqueness (Gronwall). Globalization as in [GN26, Theorem 2.5.18]: given $(\tau, t_0, y_0, p) \in \Omega$, the set of times t_1 near which Ω is a neighbourhood and Fl is regular contains t_0 (a local flow) and is open and closed in the interval of the solution: near a limit point θ, with a regular t_2 close to θ, $\mathrm{Fl}(t, t_1, y_1, q) = \Phi_\theta(t, t_2, \mathrm{Fl}(t_2, t_1, y_1, q), q)$ for the local flow Φ_θ at θ. A real analytic f is C^1 in (t, y) for fixed p, hence locally Lipschitz in y, and admits real analytic local flows by Theorem 10.11; a C^k map, $k \geq 1$, is locally Lipschitz in y by the continuity of $\mathrm{d}f$, and admits C^k local flows by the same theorem; so does a continuous map with a local Lipschitz condition. □

Chapter 11

The Frobenius theorem

The local Frobenius theorem on Banach spaces, in the form of Penot [Pen70]: a total differential equation $D_1\alpha(x,y) = f(x,\alpha(x,y))$ is integrable when it satisfies the symmetry condition (Frob), and the solution is as regular as f, including the real analytic case. In chart form this is the integrability of involutive split distributions [Lan99, Chapter VI]. Outside Banach spaces the theorem fails, already for distributions of rank one on a Fréchet space (Chapter 16).

11.1 The local Frobenius theorem on Banach spaces

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, E, F Banach spaces over $\mathbb{K}$, $O \subseteq E \times F$ open and $f\colon O \to L(E,F)$ of class C^n over $\mathbb{K}$, $1 \le n \le \infty$. We follow Penot, who derives the Frobenius theorem from the implicit function theorem [Pen70, Théorème 1.1].

Lemma 11.1 (Penot's map)**.** *The map $P(x,y,\delta) = (x, y, \delta - (s \mapsto f(sx, y + \int_0^s \delta)\, x))$ on $E \times F \times C([-1,1],F)$ is C^n where the curve $s \mapsto (sx, y + \int_0^s \delta)$ has values in O. If $(0,y_0) \in O$, its derivative at $(0,y_0,0)$ is the invertible shear $(a,b,c) \mapsto (a, b, c - f(0,y_0)\,a)$.*

Proof. P is built from continuous linear maps, the superposition operator of f (Lemma 10.3) and the continuous bilinear pointwise application $C([-1,1],L(E,F)) \times C([-1,1],E) \to C([-1,1],F)$. At $x = 0$ the term $f(\dots)\,x$ vanishes, and so does the part of its derivative that differentiates f. □

Theorem 11.2 (Local Frobenius theorem [Pen70, Théorème 1.1 and Remarque 1])**.** *Assume (Frob): for $q \in O$ and $a, b \in E$,*

$$D_1 f(q)(a)(b) + D_2 f(q)(f(q)\,a)(b) = D_1 f(q)(b)(a) + D_2 f(q)(f(q)\,b)(a).$$

For every $(x_0,y_0) \in O$ there are $r > 0$ with $B(x_0,r) \times B(y_0,r) \subseteq O$ and a C^n map $\alpha\colon B(x_0,r) \times B(y_0,r) \to F$ with $\alpha(x_0,y) = y$ and $D_1\alpha(x,y) = f(x,\alpha(x,y))$. If $\mathbb{K} = \mathbb{C}$ and f is holomorphic, α is holomorphic.

Proof. Translate x_0 to 0. By Lemma 11.1 and the inverse function theorem there is a C^n family $\varphi(x,y) \in C([-1,1],F)$ with $P(x,y,\varphi(x,y)) = (x,y,0)$ and $\varphi(0,y) = 0$; put $\alpha(x,y) = y + \int_0^1 \varphi(x,y)$. Let $c(s) = (sx, y + \int_0^s \varphi(x,y))$ and $z \in E$. Differentiating the fixed point identity, $K(s) = \partial_x(\int_0^s \varphi)(x,y)\,z$ solves the linear equation $u' = Df(c(s))(sz,u)\,x + f(c(s))\,z$ with $u(0) = 0$. By (Frob), so does $L(s) = s\,f(c(s))\,z$, and uniqueness for linear equations gives $K(1) = L(1)$, which is $D_1\alpha(x,y)\,z = f(x,\alpha(x,y))\,z$. Penot instead shows that D_3P maps K' and

L' to the same point and that it is invertible near 0; and he works with curves of class C^1 and the implicit function theorem, where we take $\delta = \gamma'$ as unknown. We do not claim this argument is new. For $\mathbb{K} = \mathbb{C}$, holomorphic maps are C^1 over $\mathbb{C}$ and C^1 maps over $\mathbb{C}$ are holomorphic (Theorem 9.9). □

Theorem 11.3 (Local Frobenius theorem, real analytic case [Pen70, Théorème 1.1])**.** *Let E, F be real Banach spaces and $f\colon O \to L(E,F)$ real analytic in the sense of Milnor, satisfying (Frob). Then the map α of Theorem 11.2 is real analytic on some $B(x_0,r)\times B(y_0,r)$.*

Proof. Penot uses a real analytic implicit function theorem; we argue with flows instead. The vector field $X(s,v,c) = (1, 0, f(x_0+sv,c)\,v)$ on $\mathbb{R}\times E\times F$ is real analytic. Since $\frac{d}{dt}\alpha(x_0+tv,y) = f(x_0+tv,\alpha(x_0+tv,y))\,v$, the curve $t\mapsto(t,v,\alpha(x_0+tv,y))$ is an integral curve of X through $(0,v,y)$, so by uniqueness it is $t\mapsto \mathrm{Fl}(t,(0,v,y))$, where Fl is the local flow of X on $(-\varepsilon,\varepsilon)\times V$. Hence $\alpha(x,y)$ is the last component of $\mathrm{Fl}(\varepsilon/2, (0, \frac{2}{\varepsilon}(x-x_0), y))$, which is real analytic by Theorem 10.6. We do not claim this argument is new. □

Proposition 11.4 (Involutivity in local form [Lan99, Chapter VI, §1, p. 157])**.** *Let E, F be normed spaces and $f\colon E\times F\to L(E,F)$ differentiable at q. If the vector fields X, Y are differentiable at q and $X(p), Y(p)\in \operatorname{graph} f(p)$ for p near q, then $[X,Y](q)\in\operatorname{graph} f(q)$ if and only if $Df(q)(X(q))(Y_1(q)) = Df(q)(Y(q))(X_1(q))$. Hence (Frob) at q holds if and only if the distribution $p\mapsto\operatorname{graph} f(p)$ is involutive at q.*

Proof. Differentiating $X_2 = f\,X_1$ gives $DX_2(q)\,v = Df(q)(v)\,X_1(q) + f(q)\,DX_1(q)\,v$, and similarly for Y. In $[X,Y](q)_2 - f(q)\,[X,Y](q)_1$ the terms with DX_1 and DY_1 cancel, leaving $Df(q)(X(q))\,Y_1(q) - Df(q)(Y(q))\,X_1(q)$. For the converse take the fields $p\mapsto(a, f(p)\,a)$ and $p\mapsto(b, f(p)\,b)$. □

Theorem 11.5 (Frobenius theorem in chart form [Lan99, Chapter VI, Theorems 1.1 and 1.2])**.** *Under the hypotheses of Theorem 11.2, every $(x_0,y_0)\in O$ is fixed by a C^n diffeomorphism $\Phi(x,y) = (x,\alpha(x,y))$ between open subsets of O with $D\Phi(p)(a,0) = (a, f(\Phi(p))\,a)$. Thus $D\Phi^{-1}(z)$ maps $\operatorname{graph} f(z)$ onto $E\times 0$: the chart Φ^{-1} rectifies the distribution. If $\mathbb{K}=\mathbb{C}$ and f is holomorphic, Φ and Φ^{-1} are holomorphic.*

Proof. Take α from Theorem 11.2. Since $D_1\alpha = f(x,\alpha)$ and $\alpha(x_0,\cdot) = \mathrm{id}$, the derivative of Φ at (x_0,y_0) is the shear $(a,b)\mapsto(a, f(x_0,y_0)\,a + b)$, which is invertible. Apply the inverse function theorem and restrict to the open set where $D\Phi$ is invertible. □

Chapter 12

Complex structures and the Newlander–Nirenberg theorem

Almost complex structures on open subsets of real Banach spaces. The Nijenhuis tensor of a real analytic almost complex structure J vanishes if and only if the distribution $\ker(J+i)$ is involutive, and then, by the Frobenius theorem in the holomorphic setting (Chapter 11), J admits holomorphic charts: this is the local Newlander–Nirenberg theorem in the real analytic case, after Penot [Pen70]. The chapter also collects what is needed about complexifications of normed spaces and about complex structures on a fixed space, which lc-manifolds uses to give the holomorphic atlas a single model.

12.1 The Nijenhuis tensor

Definition 12.1 (Nijenhuis tensor [Pen70, p. 73])**.** For a field of operators $A\colon G \to L(G)$ differentiable at z, put

$$N_A(z)(a,b) = DA(z)(Aa)\,b - DA(z)(Ab)\,a + A\,DA(z)(b)\,a - A\,DA(z)(a)\,b,$$

with $A = A(z)$. For vector fields X, Y differentiable at z, $N_A(z)(X(z),Y(z)) = ([AX,AY] + A^2[X,Y] - A[X,AY] - A[AX,Y])(z)$.

Theorem 12.2 (Vanishing Nijenhuis tensor [Pen70, Théorème 3.5, (4) $\Rightarrow$ (6)])**.** *Let G be a complex normed space and F differentiable at z with $F^2 = -1$ near z and $N_F(z) = 0$. Then $\ker(F+i)$ is involutive at z.*

Proof. Differentiating $F^2 = -1$, $DF(z)(v)$ anticommutes with $F(z)$ and maps $\ker(F+i)$ into $\ker(F-i)$. For Z, W of type $(0,1)$, $c = DF(W)Z - DF(Z)W$ lies in $\ker(F-i)$, while $N_F(Z,W) = (F+i)c$ puts it in $\ker(F+i)$; so $c = 0$. Differentiating $FZ = -iZ$ gives $(F+i)[Z,W] = c$. □

Proposition 12.3 (Complexification [Pen70, Théorème 3.5, (2) $\Rightarrow$ (4)])**.** *If $F(x+i0)$ extends $J(x)$ complex linearly for real x near x_0, then $N_F(x_0)(a+i0, b+i0) = N_J(x_0)(a,b) + i0$; hence $N_J(x_0) = 0$ implies $N_F(x_0) = 0$.*

Proof. The derivative of F in real directions extends that of J, and a complex bilinear map on $E_{\mathbb{C}}$ vanishing on real vectors vanishes. □

Theorem 12.4 (Graph form of the $(0,1)$ distribution [Pen70, p. 75])**.** *Let G be a complex Banach space and $F\colon O \to L(G)$ holomorphic with $F^2 = -1$ and $N_F = 0$. Near $z_0 \in O$, in the coordinates $G \cong \ker(F(z_0)+i) \times \ker(F(z_0)-i)$, the distribution $z \mapsto \ker(F(z)+i)$ is the field of graphs of a holomorphic map satisfying (Frob).*

Proof. With $Q = (1+iF(z_0))/2$, $P = 1-Q$ and $Q_z = (1+iF(z))/2$, the space $\ker(F(z)+i)$ is the graph of $T_z S_z^{-1}$, $S_z = QQ_z$, $T_z = PQ_z$ on $\ker(F(z_0)+i)$, as long as S_z is invertible and $\|Q_z - Q\| < 1$. (Frob) follows from Theorem 12.2 and Proposition 11.4. Penot works with the projector and his Théorème 2.3 instead; we do not claim this argument is new. □

12.2 The Newlander–Nirenberg theorem, real analytic case

Lemma 12.5 (Holomorphic extension of a real analytic field of operators)**.** *If $J\colon E \to L(E)$ is real analytic on U, there is a holomorphic $F\colon W \to L_{\mathbb{C}}(E_{\mathbb{C}})$ on an open $W \supseteq U + i0$ with $F(x+i0) = J(x)_{\mathbb{C}}$.*

Proof. Compose the complex analytic extension of J given by Milnor's definition with the continuous complex linear map $A + iB \mapsto (u+iv \mapsto (Au - Bv) + i(Av + Bu))$. □

Theorem 12.6 (Real analytic inverse function theorem)**.** *Let E, F be real Banach spaces. A real analytic map $f\colon U \to F$ (in the sense of Milnor) with $Df(x_0)$ invertible restricts to a real analytic diffeomorphism between open neighbourhoods of x_0 and $f(x_0)$; the inverse of a real analytic diffeomorphism is real analytic.*

Proof. A complex analytic extension g of f to an open subset of the normed complexification $E_{\mathbb{C}}$ has derivative $Df(x_0)_{\mathbb{C}}$ at $x_0 + i0$, which is invertible. The holomorphic inverse function theorem gives a holomorphic local inverse ψ of g, and $\psi(y+i0) = f^{-1}(y) + i0$ for real y near $f(x_0)$, since $g(f^{-1}(y)+i0) = y + i0$. So ψ extends f^{-1} near $f(x_0)$, and real analyticity is a local property [GN26, Lemma 2.2.3]. □

Theorem 12.7 (Local Newlander–Nirenberg theorem [Pen70, Théorème 3.5, (6) ⇒ (1), pp. 75–76])**.** *Let E be a real Banach space, $U \subseteq E$ open and $J\colon U \to L(E)$ real analytic with $J^2 = -1$ and $N_J = 0$. Every $x_0 \in U$ lies in the domain of a chart φ, real analytic with real analytic inverse, into the complex Banach space $\ker(J(x_0)_{\mathbb{C}} - i)$, with $D\varphi(x)\, J(x) = i\, D\varphi(x)$.*

Proof. Extend J to F (Lemma 12.5). $F^2 = -1$ and $N_F = 0$ hold at the real points (Proposition 12.3), hence on a ball around $x_0 + i0$ by the identity theorem. By Theorem 12.4 and the holomorphic Frobenius theorem (Theorem 11.5) there is Φ with $D\Phi^{-1}$ mapping $\ker(F+i)$ into $\ker(F(z_0)+i) \times 0$. The transverse coordinate $w = (\Phi^{-1} \circ \ell)_2$ kills $\ker(F+i)$; at real points, applied to $a + iJ(x)a \in \ker(F+i)$, this gives $Dw(x)(J(x)a) = i\, Dw(x)\, a$. At x_0 the kernel of Dw is $\ker(F(z_0)+i)$, which meets E only in 0, and Dw is onto, so $D(w|_E)(x_0)$ is a real isomorphism and the inverse function theorem applies. The chart $w|_E$ is real analytic since w is holomorphic, and so is its inverse (Theorem 12.6). Penot uses $x \mapsto f(x,0)$ instead; we do not claim this variant is new. □

Theorem 12.8 (Complex structures)**.** *A complex structure J ($J^2 = -1$) on a real topological vector space E makes it a complex topological vector space, $(a+ib)v = av + bJv$. A real isomorphism S with $J'S = SJ$ is a complex isomorphism $(E,J) \cong (E,J')$; on a Banach space, if $\|(J'-J)J\| < 2$ then $S = 1 - J'J$ is such an isomorphism. Finally $(E,J) \cong \ker(J_{\mathbb{C}} - i)$ by $v \mapsto (v - iJv)/2$: the models of the holomorphic charts of Theorem 12.7 at different points are isomorphic for nearby points.*

Proof. $J'S = J' + J = SJ$, and $S = 2 - (J' - J)J$ is invertible by the Neumann series. The inverse of $v \mapsto (v - iJv)/2$ is $w \mapsto 2 \operatorname{Re} w$. □

Chapter 13

Direct limits of topological spaces and k_ω-spaces

Direct limits of ascending sequences of topological spaces, groups and vector spaces, after [GN26, Appendix A.7]. They are the topological side of the locally convex direct limits of Chapter 14. The main points are that the direct limit of a compactifying sequence of Hausdorff groups is a topological group, and that k_ω-spaces (direct limits of ascending sequences of compact spaces) are stable under direct limits and finite products. Nothing in this chapter depends on the calculus.

13.1 Direct sequences and direct limits

Definition 13.1 (Direct sequences [GN26, Definitions A.7.1, A.7.5, Remark A.7.15])**.** Let $X_0 \subseteq X_1 \subseteq \cdots$ be topological spaces covering a set X, with continuous inclusions. The *direct limit topology* on X is the final topology of the inclusions. The sequence is *strict* if each X_n carries the topology induced by X_{n+1}, and *compactifying* if every point of X_n has a neighbourhood whose closure in some X_m is compact. It is *compactly regular* if every compact subset of the limit lies, and is compact, in some X_n.

Theorem 13.2 (Basic properties [GN26, Remark A.7.2, Lemmas A.7.3, A.7.6, Remark A.7.15])**.** *A set is open (closed) in the direct limit iff its traces on all steps are; a map on the limit is continuous iff its restrictions to the steps are, and this characterizes the topology; cofinal subsequences give the same limit; ascending unions of open subsets of the steps are open. If the steps are T_1, so is the limit, and compact sets lie in some step. Strict sequences have embedded steps and, for T_1 steps, are compactly regular. Locally compact Hausdorff steps are compactifying; compact Hausdorff steps are strict. Open and closed subsets carry the direct limit topology of their traces, and so do ascending unions $\bigcup_n V_n$ of open subsets $V_n \subseteq X_n$, with respect to the V_n.*

Proof. As in [GN26]. For compact sets, points $x_n \in K \setminus X_n$ would form an infinite closed discrete subset of K. For strict sequences, open subsets of X_m are extended recursively to open subsets of the later steps. □

Theorem 13.3 (Compactifying sequences [GN26, Lemmas A.7.4, A.7.6(h), A.7.9, Proposition A.7.20])**.** *Let $X_0 \subseteq X_1 \subseteq \cdots$ and $Y_0 \subseteq Y_1 \subseteq \cdots$ be compactifying sequences of Hausdorff spaces. Then $\varinjlim X_n$ is Hausdorff, compactly regular and locally k_ω, and $\varinjlim X_n \times \varinjlim Y_n = \varinjlim (X_n \times Y_n)$ as topological spaces. A Hausdorff space is k_ω iff it is the direct limit of an ascending sequence*

of compact subspaces; k_ω-spaces and locally compact spaces are locally k_ω, and locally k_ω-spaces are k-spaces [GN26, Remark A.7.19].

Proof. Compact subsets of a step have open neighbourhoods with compact closure in a later step, inside any prescribed open set. Iterating gives chains $K_j \subseteq V_j \subseteq K_{j+1}$, with V_j open and K_j compact in increasing steps, whose unions are open: two disjoint chains separate points, and products of chains fit inside open subsets of the product. A set meeting every K_j in a closed set is closed in $\bigcup_j V_j$, so compact subsets of $\bigcup_j V_j$ lie in some K_j (otherwise they contain an infinite closed discrete set); covering a compact set by finitely many compact pieces of such unions gives compact regularity. □

Theorem 13.4 (Groups and vector spaces [GN26, Proposition A.7.11, Corollary A.7.12, Example A.7.13])**.** *For an ascending sequence of topological groups with continuous inclusions, inversion and translations are continuous in the direct limit topology. If the sequence is compactifying and the steps are Hausdorff, the limit is a topological group, with the sets $\bigcup_n U_0U_1\cdots U_n$ (U_n open identity neighbourhoods of the steps) as a basis of identity neighbourhoods. Compactifying direct limits of Hausdorff (locally convex) topological vector spaces are (locally convex) topological vector spaces, and the direct limit topology is then the locally convex direct limit topology.*

Proof. Multiplication and scalar multiplication are continuous by the product theorem, the field being the constant compactifying sequence. In the basis the U_n can be taken convex, and ascending unions of convex sets are convex. A locally convex topology making the inclusions continuous is coarser than the locally convex direct limit topology, by its universal property. □

Theorem 13.5 (Further properties [GN26, Remark A.7.7, Example A.7.17, Remark A.7.21, Lemma A.7.22], [GGH10, Section 4])**.** *For locally compact Hausdorff steps, a point of an open set has nested open neighbourhoods $V_n \subseteq X_n$ with compact closures inside the set and $\overline{V_n} \subseteq V_{n+1}$. The sets V_n of Lemma A.7.6(b) form a compactifying sequence if the X_n do. A Hausdorff space is a k-space iff it is the direct limit of its compact subsets. Direct limits of (locally) k_ω-spaces are (locally) k_ω and compactly regular, finite products of (locally) k_ω-spaces are (locally) k_ω, and $\varinjlim X_n \times \varinjlim Y_n = \varinjlim(X_n \times Y_n)$ for locally k_ω steps.*

Proof. As in [GN26] and [GGH10]. For k_ω steps, the diagonal of cumulative k_ω-sequences is an ascending sequence of compact spaces with the same direct limit. For locally k_ω steps, open k_ω sets $W_n \subseteq X_n$ with $W_n \subseteq W_{n+1}$ (countable unions of open k_ω sets are k_ω) have union $\varinjlim W_n$. The product of two such limits is a k-space, and its compact subsets lie in products of compact subsets of the steps. □

Theorem 13.6 (The finite topology [GN26, Examples A.7.14, A.7.16])**.** *The direct limit topology of the finite-dimensional subspaces of a real vector space E is finer than every vector topology, and linear maps on it are continuous. If E has countable dimension, it is a Hausdorff locally convex vector topology, hence the finest vector topology and the finest locally convex topology, and $E \cong \mathbb{R}^{(\mathbb{N})}$.*

Proof. Linear maps on finite-dimensional spaces are continuous. For countable dimension, the spans of the first n basis vectors are cofinal and locally compact, so the limit is a locally convex vector topology by Corollary A.7.12. □

Chapter 14

Locally convex direct sums, direct limits and Silva spaces

Locally convex direct sums and direct limits, after [GN26, Appendix B.13 and Sections 2.7–2.8]. They are the tool for spaces of compactly supported functions and sections, fine box products of manifolds and manifolds of mappings on non-compact manifolds [GN26, Sections 4.3–4.4].

14.1 Direct sums

Definition 14.1 (Locally convex direct sum)**.** For topological vector spaces $(E_j)_{j\in J}$, $\bigoplus_j E_j$ is the space of finitely supported families with the *box topology*: the boxes $\bigoplus_j V_j = \{x \mid x_j \in V_j \text{ for all } j\}$, V_j a 0-neighbourhood in E_j, form a basis of 0-neighbourhoods.

Theorem 14.2 (Basic properties [GN26, Remark B.13.1, Lemma B.13.2(a), (c)])**.** *The box topology is a vector topology over any nontrivially normed field, locally convex and Hausdorff when the E_j are; boxes of open sets are open; coordinates and inclusions are continuous. For countable J, an additive map from $\bigoplus_j E_j$ to a topological group is continuous as soon as its restrictions to the summands are; hence the box topology is the locally convex direct sum topology of [GN26, (B.24)]. Moreover $\bigoplus_j (E_j \times F_j) \cong \bigoplus_j E_j \times \bigoplus_j F_j$ and, for any index sets, $\bigoplus_{(i,j)} E_{ij} \cong \bigoplus_i \bigoplus_j E_{ij}$ [GN26, Lemma B.13.2(b)].*

Proof. For the universal property, as in [GN26, Lemma B.13.2(a)]: choose 0-neighbourhoods Q_k with $Q_{k+1} + Q_{k+1} \subseteq Q_k$ and an injection $n\colon J \to \mathbb{N}$; the box of preimages of Q_{n_j+1} is mapped into Q_0, by induction on the support, adding the summand of least index first. For uncountable J the locally convex direct sum topology can be strictly finer than the box topology. □

Theorem 14.3 (Subspaces, bounded sets, completeness [GN26, Lemma B.13.2(d)–(f)])**.** *If $F_j \subseteq E_j$ are subspaces, $\bigoplus_j F_j$ carries the topology induced by $\bigoplus_j E_j$. If the E_j are Hausdorff, every bounded subset of $\bigoplus_j E_j$ lies in $\bigoplus_{j\in J_0} E_j$ for a finite $J_0 \subseteq J$, and $\bigoplus_j E_j$ is complete (quasi-complete, sequentially complete, Mackey complete) if every E_j is.*

Proof. The box of the sets $V_j \cap F_j$ is the trace of the box of the V_j. If a bounded set met infinitely many summands, at coordinates $x_{j_k} \neq 0$, a box omitting $k^{-1}x_{j_k}$ would not absorb it. For completeness, the coordinates of a Cauchy filter converge, and the limit has finite support by the same argument. Closed bounded sets, Cauchy sequences and Mackey–Cauchy sequences are

bounded, hence lie in a finite partial sum, where convergence is coordinatewise; the coordinates converge in the factors. □

Theorem 14.4 (Direct sums of C^k and real analytic maps [GN26, Proposition 2.8.1])**.** *Let $f_j\colon U_j \to F_j$ be C^k ($k \in \mathbb{N} \cup \{\infty\}$) on open sets $U_j \subseteq E_j$, with $f_j(0) = 0$ for all but finitely many j. Then $\bigoplus_j f_j\colon (x_j)_j \mapsto (f_j(x_j))_j$ is C^k on the open set $\bigoplus_j U_j$, with $\mathrm{d}(\bigoplus_j f_j) = (\bigoplus_j \mathrm{d}f_j) \circ \Phi$. If each f_j is real analytic and $0 \in U_j$ for all but finitely many j, then $\bigoplus_j f_j$ is real analytic.*

Proof. Induction on k, as in [GN26]: the difference quotients of $\bigoplus_j f_j$ only involve finitely many coordinates. In the real analytic case $\bigoplus_j \tilde{f}_j$, for complex analytic extensions $\tilde{f}_j$, is smooth with complex linear differentials, hence complex analytic, and extends $\bigoplus_j f_j$ through $(\bigoplus_j E_j)_{\mathbb{C}} \cong \bigoplus_j (E_j)_{\mathbb{C}}$. □

14.2 Direct limits

Theorem 14.5 (Locally convex direct limits [GN26, Definition B.13.4, Remark B.13.5])**.** *Let S_n be subspaces of a real vector space V, each with a vector topology. The seminorms of V whose restrictions to every S_n are continuous define the locally convex direct limit topology. The inclusions are continuous, a linear map to a locally convex space is continuous as soon as its restrictions to the S_n are, and an injective such map to a Hausdorff space makes the limit Hausdorff.*

Proof. For the universal property, compose with the gauge seminorms of the target. □

Remark 14.6 (Cofinal sequences [GN26, Remark B.13.10]). If a monotone family $(S_i)_{i \in I}$ of subspaces with continuous inclusions admits a cofinal sequence $(S_{i_n})_n$, then both have the same locally convex direct limit topology: apply the universal property in both directions.

Lemma 14.7 (Convex neighbourhoods [GN26, Lemma B.13.6])**.** *Let the S_n cover V.*

1. *A convex set $U \ni 0$ is a 0-neighbourhood of the direct limit if and only if $U \cap S_n$ is a 0-neighbourhood of S_n for every n.*
2. *For an ascending sequence with continuous inclusions, a convex set is open if and only if its traces on the S_n are open.*
3. *For countably many steps, the summation map $\bigoplus_n S_n \to V$ is an open quotient map; hence the sets $\bigcup_n (W_0 + \cdots + W_n)$, W_n (open) 0-neighbourhoods of S_n, form a basis of (open) 0-neighbourhoods.*
4. *If the S_n are barrelled, so is the direct limit.*

Proof. (a) The gauge of $U \cap (-U)$ is a seminorm on V whose restrictions to the S_n are continuous. (b) Translate to a point of U and apply (a). (c) The summation map is continuous by the universal property of the direct sum; on the quotient of the direct sum by its kernel, a locally convex space, the inclusions of the S_n are continuous, so the inverse of the induced bijection is continuous by the universal property of the direct limit. (d) A lower semicontinuous seminorm on the limit restricts to continuous seminorms on the steps. □

Lemma 14.8 (Extension of absolutely convex open sets [GN26, Lemma B.13.8])**.** *Let $j\colon F \to E$ be an injective linear topological embedding into a locally convex space. Every absolutely convex open set $V \subseteq F$ is $j^{-1}(Q)$ for an absolutely convex open set $Q \subseteq E$.*

Proof. As in [GN26], Q is the convex join of $j(V)$ and an absolutely convex open 0-neighbourhood W of E with $j^{-1}(W) \subseteq V$. A point $a\,j(v)+bw$ of Q has the open neighbourhood $a\,j(v)+bW \subseteq Q$ if $b > 0$, and $j(v) = r^{-1}j(rv) + (1-r^{-1}) \cdot 0$ with $rv \in V$, $r > 1$, otherwise. □

Theorem 14.9 (Strict direct limits [GN26, Proposition B.13.9(a)–(d)])**.** *Let $S_0 \subseteq S_1 \subseteq \cdots$ be locally convex subspaces covering V, each S_n carrying the topology induced by S_{n+1}. Then:*

1. *the direct limit induces the given topology on each S_n;*
2. *the direct limit is Hausdorff if the S_n are;*
3. *if each S_n is closed in S_{n+1}, every bounded (compact) subset of the limit is contained, and bounded (compact), in some S_n;*
4. *the direct limit is complete if the S_n are;*
5. *if each S_n is closed in S_{n+1}, the direct limit is quasi-complete (sequentially complete, Mackey complete) if the S_n are.*

Proof. (a) Extend an absolutely convex open 0-neighbourhood of S_m step by step with Lemma 14.8 and take the union, a 0-neighbourhood by Lemma 14.7. (b) follows from (a). (c) If a bounded set B met infinitely many differences $S_{n+1} \setminus S_n$, at points x_n, the extensions in (a) can be cut down so that $x_n \notin (n+1)W$, and W does not absorb B. (d) As in [GN26], a Cauchy filter concentrates near some S_m: otherwise the convex hull of sets $W_n \cap S_n$, with W_n shrinking 0-neighbourhoods, contradicts the Cauchy property. The traces on S_m of the sets $A + W$ then form a Cauchy filter on S_m, by (a), whose limit is the limit of the filter. (e) Closed bounded sets, Cauchy sequences and Mackey–Cauchy sequences (together with their bounded set) are bounded, hence lie and are bounded in some S_n, by (c); conclude with (a). □

14.3 Silva spaces

Definition 14.10 (Silva systems [GN26, Definition B.13.12])**.** A *Silva system* presenting a locally convex space E consists of continuous linear maps $j_n \colon E_n \to E$ from locally convex spaces and compact operators $u_n \colon E_n \to E_{n+1}$ with $j_{n+1} \circ u_n = j_n$, such that the ranges of the j_n cover E and a convex set $C \ni 0$ is a 0-neighbourhood as soon as every $j_n^{-1}(C)$ is. Silva spaces, locally convex direct limits of Banach spaces with compact inclusions, are the case of injective j_n (by Lemma 14.7).

Theorem 14.11 (Topology of Silva spaces [GN26, Proposition B.13.13(a)–(d)])**.** *Let E be presented by a Silva system.*

1. *A set $O \subseteq E$ is open as soon as every $j_n^{-1}(O)$ is open: the locally convex direct limit topology is the direct limit topology of the topological spaces E_n. Hence a map from E to a topological space is continuous as soon as its composites with the j_n are, and E is Hausdorff if the E_n are T_1 and the j_n injective.*
2. *Products of Silva systems are Silva systems.*

For a Silva space with T_1 sequential steps (for instance Banach spaces), compact subsets are contained, and compact, in some step; if moreover the steps have bounded 0-neighbourhoods, bounded subsets are contained, and bounded, in some step, and they are relatively compact.

Proof. We do not follow the route of [GN26] through direct limits of compactifying sequences and k_ω-spaces (Corollary A.7.12, Proposition A.7.20). (a) Given $x = j_m(y_0) \in O$, build compact sets $K_i \subseteq E_{m+i}$ with $j_{m+i}(K_i) \subseteq O$ and convex sets $A_i \ni 0$ with $x + A_i \subseteq j_{m+i}(K_i)$: if U_i is a convex 0-neighbourhood of E_{m+i} with $u_{m+i}(U_i)$ inside a compact set L_i, take $A_{i+1} = A_i + \varepsilon_i j_{m+i}(U_i)$ and $K_{i+1} = u_{m+i}(K_i) + \varepsilon_i L_i$, with $\varepsilon_i > 0$ so small that K_{i+1} is still mapped into O, by compactness. The union of the A_i is a convex 0-neighbourhood, and x plus it lies in O. (b) A convex set $C \ni 0$ contains $\frac{1}{2}C_1 \times \frac{1}{2}C_2$, where $C_1 \times \{0\}$ and $\{0\} \times C_2$ are its traces on the factors. For compact regularity, the compact sets kL_p cover E and, by (a), detect closed sets; a compact set meeting the complements of all their finite unions would contain an infinite closed discrete set. Bounded regularity and relative compactness then follow as in [GN26], with an increasing sequence of bounded 0-neighbourhoods of the steps in place of normalized unit balls. We do not claim these arguments are new. □

Theorem 14.12 (Duality for Silva spaces [GN26, Proposition B.13.13(e)–(g)])**.** *A barrelled Hausdorff locally convex space whose bounded sets are relatively compact is reflexive. Consequently, a Silva space with T_1 sequential barrelled steps having bounded 0-neighbourhoods (for instance Banach spaces) is a reflexive, complete k_ω-space, and its strong dual is a Fréchet space.*

Proof. A functional φ on the strong dual is bounded by 1 on the functionals bounded by 1 on a compact absolutely convex set C. If no point of C matched φ on finitely many functionals $f_1, \dots, f_k$, a functional on $\mathbb{R}^k$ would separate $(\varphi(f_i))_i$ from the image of C; compactness of C then gives a point matching φ everywhere. The evaluation map is continuous by Banach–Steinhaus and open by the bipolar argument (Hahn–Banach). For Silva spaces, the compact sets kL_p detect closed sets and absorb the compact sets, so E is k_ω; its topology is coherent with the bounded sets, so the strong dual is complete, and a sequence of compact sets absorbing the bounded sets makes it metrizable. The strong bidual of a metrizable space is complete, so E is complete. We do not claim this argument is new. □

Theorem 14.13 (C^k maps on Silva spaces [GN26, Proposition 2.7.1])**.** *Let E be presented by a Silva system with Hausdorff steps and $U \subseteq E$ open. A map $f \colon U \to F$ to a Hausdorff locally convex space is C^k as soon as every $f \circ j_n$ is C^k on $j_n^{-1}(U)$. In a Silva space, the union of ascending open subsets U_n of the steps is open, and a map on it is C^k as soon as its restrictions to the U_n are.*

Proof. Induction on k, as in [GN26]: the directional derivatives of f are those of the $f \circ j_n$, and $\mathrm{d}f$ is C^{k-1} on $U \times E$, presented by the product Silva system, by the induction hypothesis. Continuity comes from Theorem 14.11(a). □

Chapter 15

Krein spaces

Indefinite inner product spaces after Bognár [Bog74]: ortho-complemented subspaces of a Krein space (Chapter V), the majorant topology and the intrinsic dimension (Chapter IV), and continuous families of Krein spaces. This is the linear analysis behind the pseudo-Riemannian metrics of infinite index (Krein type) in lc-manifolds.

15.1 Complexification of real Hilbert spaces

Definition 15.1 (Complexification of an inner product space)**.** For a real (or complex, regarded as real) inner product space E, the complexification $E_{\mathbb{C}} = E \times E$, with (x, y) standing for $x + iy$, is a complex inner product space with $\langle (x,y),(u,v)\rangle = \langle x,u\rangle + \langle y,v\rangle + i(\langle x,v\rangle - \langle y,u\rangle)$ (real parts of the inner products of E), and $\|(x,y)\|^2 = \|x\|^2 + \|y\|^2$; it is complete when E is. A continuous linear operator T extends to $T_{\mathbb{C}}(x,y) = (Tx, Ty)$; $T \mapsto T_{\mathbb{C}}$ is multiplicative and preserves adjoints, and the conjugation $(x,y) \mapsto (x,-y)$ commutes with every $T_{\mathbb{C}}$.

This is the standard device that brings the continuous functional calculus of the complex C*-algebra of operators of $E_{\mathbb{C}}$ to bear on real Hilbert spaces: it gives the spectral projections and the positive and negative parts used below over $\mathbb{R}$. Mathlib has no complexification of inner product spaces in the version we use.

15.2 Ortho-complemented subspaces

Definition 15.2 (Krein spaces, fundamental symmetries, ortho-complemented subspaces)**.** Following [Bog74, §I.11, §III.8, §V.1], a subspace L of a space with a Hermitian form $[\cdot,\cdot]$ is a *Krein space* if it admits a fundamental decomposition $L = L_+ \oplus L_-$, a $[\cdot,\cdot]$-orthogonal direct sum with $[\cdot,\cdot]$ positive definite on L_+ and negative definite on L_-, whose components are *intrinsically complete*: complete for the norms $\sqrt{[x,x]}$ on L_+ and $\sqrt{-[x,x]}$ on L_-. No topology enters. The ambient space is a Hilbert space H over $\mathbb{R}$ or $\mathbb{C}$ with a *fundamental symmetry* J, a self-adjoint involution, and $[x,y] = \langle Jx, y\rangle$; every Krein space is of this form [Bog74, Corollary V.1.2]. A subspace $L \subseteq H$ is *ortho-complemented* if $L + L^{[\perp]} = H$ [Bog74, §I.9, p. 18] (*projectively complete* in [AI89, Definition 1.7.8]).

Theorem 15.3 (Ortho-complemented subspaces of a Krein space [Bog74, Theorem V.3.4])**.** *Let H be a Hilbert space with a fundamental symmetry J, and $L \subseteq H$ a subspace.*

1. *If L is ortho-complemented, it is closed and nondegenerate [Bog74, Corollaries III.2.5 and I.9.5].*

2. *L is ortho-complemented if and only if there is a continuous J-orthogonal projector Q onto L, i.e. $[Qw, t] = [w, t]$ for $t \in L$; such a projector is unique. For closed L this happens if and only if the Gram operator $G_L = P_L J|_L$ is bijective.*

3. *A closed subspace which is a Krein space with the restricted form is ortho-complemented.*

4. *A subspace is ortho-complemented if and only if it is closed and a Krein space (over $\mathbb{R}$ and $\mathbb{C}$).*

Proof. Items 1 and 2 use no completeness beyond the orthogonal projection onto L. For 3, Bognár goes through Hilbert majorants and majorant topologies [Bog74, Theorem V.1.3]; here the closed graph theorem shows that the intrinsic norms dominate the norm of H, and the Riesz representation theorem in L_+ and L_- produces the J-orthogonal projection, the route c $\Rightarrow$ d $\Rightarrow$ a of [AI89, Theorem 1.7.16]. For 4, $L_\pm$ are the spectral subspaces of G_L for $(0, \infty)$ and $(-\infty, 0)$, obtained from the continuous functional calculus in the C*-algebra $\mathcal{L}(L)$; they are closed with $\pm[x, x] \geq c\|x\|^2$, hence intrinsically complete. Over $\mathbb{R}$ the continuous functional calculus is applied to the complexification $G_{\mathbb{C}}$ of G_L on $L \times L$: its spectral projection P commutes with $G_{\mathbb{C}}$, hence with every operator commuting with $G_{\mathbb{C}}$; the conjugate CPC (for the conjugation $C(x, y) = (x, -y)$) is an idempotent commuting with P with the same sign bounds, so $CPC = P$ and P restricts to a real projection on L. We do not claim this argument is new. □

Proposition 15.4 (Positive definite subspaces)**.** *A positive definite subspace L of the Krein space H is ortho-complemented if and only if it is closed and intrinsically complete, if and only if it is closed and uniformly positive: $[x, x] \geq c\|x\|^2$ on L for some $c > 0$. A closed positive definite subspace which is not uniformly positive is nondegenerate and decomposable ($L = L \oplus 0$) but neither intrinsically complete, nor a Krein space, nor ortho-complemented.*

Proof. A Krein decomposition of a positive definite L has $L_- = 0$, so Theorem 15.3 gives the first equivalence. If L is closed and intrinsically complete, the identity of L onto its intrinsic Hilbert space is continuous ($[x, x] \leq \|J\|\|x\|^2$) and bijective between complete spaces, so its inverse is bounded by the open mapping theorem; the converse is a Cauchy sequence argument. Uses the completeness of H. □

Example 15.5 ([Bog74, Example V.3.3]). On ℓ^2 (over $\mathbb{R}$ or $\mathbb{C}$) let $Jx = (s_i x_i)$ with $s_i = -1$ for even and $s_i = 1$ for odd i, and $L = \{x \mid x_{2k+1} = \frac{2k+2}{2k+1} x_{2k}\}$, the subspace of [Bog74, Example I.9.7, pp. 19–20]. Then L is closed, positive definite, nondegenerate and decomposable, but not intrinsically complete, not a Krein space, and not ortho-complemented.

Proof. For $x \in L$, $[x, x] = \sum_k (d_k^2 - 1)|x_{2k}|^2$ with $d_k = \frac{2k+2}{2k+1}$, which is positive for $x \neq 0$, while $d_k^2 - 1 \leq 3/(2k+1) \to 0$; apply Proposition 15.4. Bognár exhibits instead an intrinsic Cauchy sequence whose would-be limit is not in ℓ^2. □

Lemma 15.6 (Sign decomposition of a self-adjoint operator)**.** *Let G be a bounded self-adjoint operator on a real or complex Hilbert space E. Then $G = G^+ - G^-$ with $G^\pm$ positive and $G^+G^- = G^-G^+ = 0$, and $E = N_0 \oplus N_+ \oplus N_-$ with $N_0 = \ker G$, $N_+ = (\ker G^+)^\perp$, $N_- = (\ker G^-)^\perp$; the three components and the sums of any two of them are closed, $\langle Gx, x\rangle$ is positive definite on N_+ and negative definite on N_-, and $\langle GN_+, N_-\rangle = 0$.*

Proof. Over $\mathbb{C}$, $G^{\pm}$ are the positive and negative parts from the continuous functional calculus. Over $\mathbb{R}$, take those of the complexification $G_{\mathbb{C}}$; $CG^{\pm}_{\mathbb{C}}C$ is again a decomposition of $G_{\mathbb{C}}$ into positive operators with product 0, so it equals $G^{\pm}_{\mathbb{C}}$ by uniqueness, and $G^{\pm}_{\mathbb{C}}$ restrict to E. Since $G^+G^- = 0$, $G^-y \in \ker G^+$, so $\langle y, G^-y\rangle = 0$ for $y \in N_+$, whence $G^-y = 0$ (a positive operator vanishes where its quadratic form does): $N_+ \subseteq \ker G^-$, and likewise $N_- \subseteq \ker G^+$. On N_+, $\langle Gx, x\rangle = \langle G^+x, x\rangle > 0$ for $x \neq 0$, since $G^+x = 0$ would put x in $\ker G^+ \cap N_+ = 0$. Decomposing along $\ker G^+$ and then $\ker G^-$ gives $E = N_0 + N_+ + N_-$, with $N_0 + N_+ = \ker G^-$, $N_0 + N_- = \ker G^+$, $N_+ + N_- = N_0^{\perp}$. These are the spectral subspaces of G for $\{0\}$, $(0,\infty)$, $(-\infty, 0)$, but no spectral measure is used. □

Theorem 15.7 ([Bog74, Theorem V.3.1])**.** *Every closed subspace L of the Krein space H has a fundamental decomposition $L = L^0 \dotplus L_+ \dotplus L_-$ [Bog74, §I.11, p. 24] (L^0 the isotropic part, L_+ positive definite, L_- negative definite, mutually orthogonal) whose three components and the sums of any two of them are closed. In particular a closed nondegenerate subspace has a fundamental decomposition $L = L_+ \oplus L_-$ with closed components.*

Proof. Apply Lemma 15.6 to the Gram operator G_L on the Hilbert space L; $\ker G_L$ is the isotropic part of L. Bognár deduces the theorem from his Theorem IV.5.2 (p. 89), whose proof uses the spectral function of the Gram operator; since 0 may lie in the spectrum of G_L, the corresponding spectral projections are not continuous functions of G_L, and Lemma 15.6 replaces them. □

Theorem 15.8 ([Bog74, Theorem V.3.5])**.** *A subspace L of the Krein space H is ortho-complemented if and only if it is closed, nondegenerate, and every fundamental decomposition $L = L_+ \oplus L_-$ extends to a fundamental decomposition $H = H_+ \oplus H_-$ with $L_+ \subseteq H_+$ and $L_- \subseteq H_-$.*

Proof. "Only if": L is closed and nondegenerate by Theorem 15.3, and $L^{[\perp]}$ is ortho-complemented, hence a Krein space with a decomposition $M_+ \oplus M_-$; $H_\pm = L_\pm + M_\pm$, as in Bognár. "If": L has a fundamental decomposition by Theorem 15.7. Every fundamental decomposition of H has uniformly definite components (they are ortho-complemented and definite, Proposition 15.4); $L_+ = L \cap L_-^{[\perp]}$ is closed and, inside H_+, uniformly positive, hence ortho-complemented; likewise L_-; and an orthogonal sum of ortho-complemented subspaces is ortho-complemented [Bog74, Lemma I.9.2]. Bognár instead ortho-complements L_+ inside the Hilbert space $(H_+, [\cdot,\cdot])$. We do not claim this argument is new. □

15.3 The majorant topology

Definition 15.9 (J-inner product of a fundamental decomposition)**.** For a fundamental decomposition $E = E_+ \oplus E_-$ of a Hermitian form B, with projections $P_\pm$ and fundamental symmetry $J = P_+ - P_-$, the *J-inner product* $[x,y]_J = B(Jx, y) = B(P_+x, P_+y) - B(P_-x, P_-y)$ is positive definite. Its norm defines the *decomposition majorant* [Bog74, §IV.5–6].

Theorem 15.10 (All decomposition majorants coincide [Bog74, Theorem IV.6.4, Corollary IV.6.3, Theorem IV.7.2])**.** *Let (E, B) be a Krein space and K a Krein decomposition. Then E with $[\cdot,\cdot]_K$ is a Hilbert space, every fundamental decomposition D of (E, B) has a J-norm equivalent to that of K, and D is a Krein decomposition. On a Hilbert space with $B = \langle G\cdot, \cdot\rangle$, G invertible and self-adjoint, every fundamental decomposition has closed components, a continuous fundamental symmetry, and a J-norm equivalent to the norm.*

Proof. $E_\pm = E_\mp^{[\perp]}$ is closed, so J is continuous by the closed graph theorem. The operator $U = GJ$ is self-adjoint, positive ($\langle Ux, x\rangle = [x,x]_J$) and invertible, so $\|Ux\|^2 \leq \|U\|\langle Ux, x\rangle$

(Cauchy–Schwarz for the positive form) and $\|x\| \le \|U^{-1}\|\|Ux\|$ give $c\|x\|^2 \le [x,x]_J \le \|U\|\|x\|^2$. The algebraic case is reduced to this one on the Hilbert space of K, whose Gram operator is J_K. Bognár proves Theorem 6.4 for an arbitrary Hilbert majorant by an iteration; for Krein spaces the positivity and invertibility of GJ suffice. We do not claim this argument is new. □

15.4 The intrinsic dimension

Theorem 15.11 (Rank of positivity and of negativity [Bog74, Lemma IV.7.1, Corollary IV.7.4])**.** *The* dense-span dimension *of a topological vector space is the least cardinality of a set with dense span; the* intrinsic dimension *of a definite subspace is its dense-span dimension for the intrinsic norm. For two fundamental decompositions of a Krein space, $P_{1+}\colon E_{2+} \to E_{1+}$ is an isomorphism of topological vector spaces, so the intrinsic dimensions of E_+ and of E_- (the index) do not depend on the decomposition.*

Proof. For $x \in E_{2+}$, $B(x,x) = B(P_{1+}x, P_{1+}x) + B(P_{1-}x, P_{1-}x) \le B(P_{1+}x, P_{1+}x)$, so P_{1+} is bounded below; it is bounded by Theorem 15.10. Its range is closed, and its orthogonal in E_{1+} lies in $E_{2+}^{[\perp]} = E_{2-}$, hence is zero. Bognár deduces Corollary 7.4 from his Lemma 7.3 instead. □

Theorem 15.12 (Dense-span dimension and Hilbert dimension)**.** *In an inner product space, no orthonormal family is larger than a set with dense span. Hence the dense-span dimension of an inner product space is the cardinality of any orthonormal family with dense span; in particular all Hilbert bases of a Hilbert space have the same cardinality, and the ranks of positivity and of negativity of a Krein space are the Hilbert dimensions of E_+ and E_-.*

Proof. Let (v_i) be orthonormal and s have dense span. Each v_i has a nonzero inner product with some element of s, otherwise v_i would be orthogonal to the closure of the span of s. If s is infinite, each element of s has nonzero inner products with countably many v_i only (Bessel's inequality), so the index set has cardinality at most $|s| \cdot \aleph_0 = |s|$. If s is finite, its span is closed, hence everything, and the v_i are linearly independent. □

15.5 Continuous families of Krein spaces

Theorem 15.13 (Continuous spectral projections and local constancy of the index)**.** *Let $G(x)$ be a norm-continuous family of invertible self-adjoint operators on a real or complex Hilbert space. The spectral projections of $G(x)$ onto the positive part depend continuously on x, so fundamental decompositions of $(H, \langle G(x)\cdot,\cdot\rangle)$ can be chosen continuously; and the index of any choice of fundamental decompositions is a locally constant function of x.*

Proof. Over $\mathbb{C}$ the projection is $\chi_{(0,\infty)}(G(x))$, and the continuous functional calculus is continuous in the operator when the spectra stay in a fixed open set on which the function is continuous (upper hemicontinuity of the spectrum); over $\mathbb{R}$ use the complexification. If P, Q are orthogonal projections with $\|P - Q\| < 1$, then $v \mapsto v - Qv$ is an isomorphism $\ker P \cong \ker Q$. The index is computed on the spectral decomposition, where the intrinsic norm on $\ker P$ is equivalent to the norm. □

Chapter 16

Counterexamples

Many theorems of the Banach theory fail on Fréchet spaces, and most of the hypotheses in this document are there because of an example. This chapter collects the examples that are formalized. Each one sits next to the positive result it limits, which is also stated in Lean, so that the boundary between the two settings can be read off from the formal statements. Two further examples are stated where they belong: the counterexamples for series of homogeneous polynomials (Remark 9.27) and Bognár's closed positive definite subspace of a Krein space (Example 15.5).

Throughout, s denotes the Fréchet space of rapidly decreasing real sequences, with the seminorms $p_k(x) = \sup_n (n+1)^k |x_n|$, and $\mathbb{R}^{\mathbb{N}}$ the Fréchet space of all real sequences with the product topology.

16.1 Algebras

Example 16.1 (Units need not form an open set [GN26, Example 5.3.9(b)]). In a complete normed ring the units form an open set. In the complete locally convex algebra $C(\mathbb{R}, \mathbb{R})$, with the compact-open topology, they do not: the plateau functions, equal to 1 on $[-n, n]$ and to 0 outside $(-n-1, n+1)$, are not units, and they converge to 1.

16.2 Inverse functions

Example 16.2 (The inverse function theorem fails on $\mathbb{R}^{\mathbb{N}}$, after [Ham82, Counterexample 5.5.1]). On a Banach space, a map strictly differentiable at a with derivative the identity sends neighbourhoods of a onto neighbourhoods of $f(a)$. On $\mathbb{R}^{\mathbb{N}}$ the map $P(a)_k = a_k - \sum_{j \le k} j\, a_{k-j} a_j$ is smooth, with $P(0) = 0$ and $\mathrm{d}P(0) = \mathrm{id}$, but its image is not a neighbourhood of 0: the sequence $g_n = \frac{1}{n} e_0 + e_n$ tends to 0, and $g_n \notin P(\mathbb{R}^{\mathbb{N}})$ for $n \ge 2$.

Remark 16.3. Hamilton's operator is $Pf = f - xff'$ on $C^\infty[-1, 1]$. The map above is its action on the coefficients of formal power series at 0, which is where his argument takes place; we state it on $\mathbb{R}^{\mathbb{N}}$ to avoid power series.

Example 16.4 (Lower bounds are not locally uniform). On normed spaces, if $(y, v) \mapsto L(y)v$ is C^1 and $L(y_0)$ is a topological embedding, then the $L(y)$ are bounded below, with a common constant, for y near y_0. On s there is a smooth family $L(t)$ of linear automorphisms, $L(t)x = (a_n(t)x_n)_n$, with $L(0) = \mathrm{id}$, whose lower bound is not locally uniform near $t = 0$: the inverses $L(t)^{-1}$ are not equicontinuous on any neighbourhood of 0. This is why the hypothesis of Definition 8.8 is needed.

Example 16.5 (Invertible derivatives do not reflect smoothness). There are a smooth bijection j of $\mathbb{R} \times s$, whose derivative at every point is a continuous linear automorphism, and a continuous curve γ such that $j \circ \gamma$ is smooth but γ is not C^1 on any neighbourhood of 0. Moreover j satisfies the hypothesis of Theorem 8.12 at each single point but not near every point, $(p, v) \mapsto \mathrm{d}j(p)^{-1}v$ is not continuous, and j^{-1} is not C^1 near $(0, c)$. Compare Theorem 8.6 and Theorem 8.12.

16.3 Differential equations

Example 16.6 (Uniqueness fails for linear equations [GN26, Example 2.4.3]). On a normed space, the only solution of $y' = Ay$, $y(0) = 0$, for a continuous linear A, is $y = 0$ [GN26, Proposition 2.4.11]. On $\mathbb{R}^{\mathbb{N}}$, with A the left shift, the curve $t \mapsto (h(t), h'(t), h''(t), \dots)$ is a nonzero solution with $y(0) = 0$, where h is the smooth function $h(t) = e^{-1/t}$ for $t > 0$, $h(t) = 0$ for $t \le 0$, all of whose derivatives vanish at 0.

Example 16.7 (Existence fails for linear equations). On a Banach space, $t \mapsto e^{tA}y_0$ solves $y' = Ay$, $y(0) = y_0$, for all t. On s, with $Dx = (n^2 x_n)_n$, the equation $y' = Dy$ has no solution on any interval $[0, \varepsilon)$ with $y(0) = (e^{-(n+1)})_n$. This is the heat equation with reversed time on $C^\infty(S^1)$, written on the Fourier side; the example goes back to Milnor [Mil84, §3, exercise after Warning 3.6, p. 1025], and is presented in [Nee06, Example II.3.12] (numbering of the arXiv version).

Example 16.8 (The Frobenius theorem fails on s [KM97, 32.12, Example 1]). There are a continuous linear operator D of s and a point a with $Da \neq 0$ such that no curve through a is tangent to the rank one distribution $x \mapsto \mathbb{R}\, Dx$; compare Theorem 11.2.

16.4 Analytic maps

Example 16.9 (Complex analytic maps need not be locally bounded [GN26, Exercise 2.1.10]). A complex analytic map into a normed space is bounded near each point. The map $f(x) = \sum_k 2^{k+1} x_k^{2k+2}$ is complex analytic on $\ell^1(\mathbb{N}, \mathbb{C})$ but unbounded on the unit ball, and $g(x) = (f((k+1)x))_k$ is a complex analytic map from ℓ^1 to the Fréchet space $\mathbb{C}^{\mathbb{N}}$ which is bounded on no neighbourhood of 0.

16.5 Symplectic forms

Example 16.10 (Darboux's theorem fails for weak symplectic forms [Mar72]). By the Darboux–Weinstein theorem a strong symplectic form on a Banach space is constant in suitable charts [Wei69] (formalized in lc-manifolds). On $H = \ell^2(\mathbb{N}, \mathbb{R})$, with $S = \mathrm{diag}(1/(n+1))$, Marsden's form $\Omega = -\mathrm{d}\theta$ on $H \times H$, where $\theta_{(x,e)}(e_1, e_2) = g_x(e, e_1)$ and $g_x(e, f) = \langle (S + \|x\|^2 I)e, f\rangle$, is smooth, closed and weakly nondegenerate, but it admits no Darboux chart at the origin; it cannot even be made constant by a vector bundle chart of the fibres, as observed by the referee of [Mar72].

Proof. Ω is strongly nondegenerate at $(x, 0)$ for $x \neq 0$, since $S + \|x\|^2 I$ is onto, and not at $(0, 0)$, since S is not onto. A chart carrying Ω to a constant form at every point would make both nondegeneracy conditions equivalent. □

Chapter 17

Limitations of the Michal–Bastiani calculus

The Michal–Bastiani calculus is defined on arbitrary locally convex spaces, needs no completeness, and its smooth maps are continuous. The price is paid elsewhere: spaces of operators and of smooth maps carry several natural topologies, none of which makes evaluation continuous in infinite dimensions, and as a consequence the category of smooth maps is *not* cartesian closed. This chapter collects these caveats, explains why they are not a matter of completeness, and records the convention that we follow throughout to avoid them: operator-valued data are always required to be smooth *jointly* (in uncurried form), never as maps into a space of operators. Nothing in this chapter is formalized; it is here to explain the design.

Throughout, E and F are locally convex spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and $\mathcal{L}(E, F)$ is the space of continuous linear maps.

17.1 Topologies on spaces of operators

Three topologies on $\mathcal{L}(E, F)$ will appear, each one the topology of uniform convergence on a family of subsets of E:

- $\mathcal{L}(E, F)_s$: uniform convergence on finite sets (pointwise convergence); on normed spaces this is the *strong operator topology*;
- $\mathcal{L}(E, F)_c$: uniform convergence on compact sets (the compact-open topology);
- $\mathcal{L}(E, F)_b$: uniform convergence on bounded sets; on normed spaces this is the *operator norm* topology, since it suffices to look at the unit ball.

Since finite sets are compact and compact sets are bounded, $s \subseteq c \subseteq b$.

A word of caution on terminology: in operator theory "strong topology" means s, while in the theory of locally convex spaces the "strong topology" (on a dual space, say) is b.

Remark 17.1 (s and c agree on equicontinuous sets). On an equicontinuous subset $H \subseteq \mathcal{L}(E, F)$ (for normed E, F: a set bounded in operator norm) the topologies s and c coincide. Indeed, given a compact $K \subseteq E$, a continuous seminorm q on F and $\varepsilon > 0$, equicontinuity gives a 0-neighborhood V with $q(Tv) < \varepsilon$ for $T \in H$, $v \in V$; covering K by finitely many translates $x_i + V$ reduces uniform convergence on K to convergence at the finitely many points x_i. In particular,

for a Banach space E, a sequence in $\mathcal{L}(E,F)$ which converges in s is bounded in operator norm (uniform boundedness principle), hence converges in c: on Banach spaces the two topologies can only be distinguished by nets.

Remark 17.2 (c is strictly finer than s in infinite dimensions). If E is infinite-dimensional and $F \neq 0$, then $s \neq c$ on $\mathcal{L}(E,F)$. It suffices to treat $F = \mathbb{K}$, i.e. the dual E'. Every s-neighborhood of 0 in E' contains the annihilator of a finite set $\{x_1, \dots, x_k\}$. If the polar K° of a compact set K contained such an annihilator, then every $f \in E'$ vanishing on the x_j would vanish on K (scale f), and by the Hahn–Banach theorem K would lie in the finite-dimensional span of the x_j. A compact set not contained in any finite-dimensional subspace exists (for normed E: $\{0\} \cup \{x_n/n\}$ with x_n linearly independent unit vectors), so K° is a c-neighborhood of 0 which is not an s-neighborhood.

Example 17.3.

1. (Incomplete E, sequences.) Let $E = c_{00}$ be the space of finitely supported sequences with the supremum norm, and $\varphi_n = n\, e_n^*$. For each $x \in E$, $\varphi_n(x) = 0$ for large n, so $\varphi_n \to 0$ in s. On the compact set $K = \{0\} \cup \{e_m/m : m \geq 1\}$ we have $\varphi_n(e_n/n) = 1$, so $\varphi_n \not\to 0$ in c. (By Remark 17.1, such a sequence cannot exist on a Banach space.)
2. ($E = \ell^2$, nets.) Let $A = \{\sqrt{n}\, e_n^* : n \geq 1\} \subseteq (\ell^2)'$. Then 0 lies in the s-closure of A: otherwise there would be $x^{(1)}, \dots, x^{(k)} \in \ell^2$ and $\varepsilon > 0$ with $\max_j \sqrt{n}\, |x_n^{(j)}| \geq \varepsilon$ for all n, whence $\sum_j |x_n^{(j)}|^2 \geq \varepsilon^2/n$, and summing over n gives a contradiction. But 0 is not in the c-closure of A: on the compact set $K = \{0\} \cup \{e_m/\sqrt{m}\}$ each element of A takes the value 1.

For a Banach space E, the topology c on E' is the *bounded weak-$*$ topology*: the finest topology which agrees with the weak-$*$ topology on bounded sets (Banach–Dieudonné theorem; for the statement on convex sets see [FHHMPZ01, Thm. 4.44]).

17.2 The exponential law

For locally convex subsets $U \subseteq E_1$ and $V \subseteq E_2$ with dense interior, endow $C^\infty(V,F)$ with the compact-open C^∞-topology, and write $f^\vee(x) = f(x,\cdot)$.

Theorem 17.4 ([GN26, Thm. 1.7.30, Cor. 1.7.35])**.** *For all locally convex E_1, E_2, F and every $f \in C^\infty(U \times V, F)$, the map $f^\vee \colon U \to C^\infty(V,F)$ is smooth, and $\Phi \colon C^\infty(U \times V, F) \to C^\infty(U, C^\infty(V,F))$, $f \mapsto f^\vee$, is linear and a topological embedding. If moreover V is locally compact, or $U \times V \times E_1 \times E_2$ is a $k_{\mathbb{R}}$-space, then Φ is an isomorphism of topological vector spaces.*

Remark 17.5 (Completeness plays no role). No completeness is assumed in Theorem 17.4. The hypothesis for the converse direction is topological: a $k_{\mathbb{R}}$-space is a Hausdorff space on which a real function is continuous as soon as its restrictions to compact sets are [GN26, Rem. A.6.15]. It holds when E_1 and E_2 are metrizable (normed spaces, complete or not; Fréchet spaces), when V is finite-dimensional, and also for k_ω-spaces such as countable direct limits of finite-dimensional spaces, e.g. $\mathbb{R}^{(\mathbb{N})}$ [GN26, Rem. 4.1.22, Prop. A.7.20]. So the absence of a countable basis of 0-neighborhoods only threatens the converse direction, and only for spaces which are not $k_{\mathbb{R}}$-spaces.

17.3 Evaluation and cartesian closedness

The category would be cartesian closed if $C^\infty(V,F)$ were an object and the evaluation $\varepsilon \colon C^\infty(V,F) \times V \to F$ were smooth. This holds when V is finite-dimensional: then ε is $C^{\infty,\ell}$ [GN26, Lemma 1.7.25]. It fails in every infinite-dimensional case, and already at the linear level.

Proposition 17.6 (Evaluation on the compact-open dual)**.** *If E is infinite-dimensional, the evaluation $E'_c \times E \to \mathbb{K}$, $(f,x) \mapsto f(x)$, is not continuous. Consequently, for every open $U \subseteq E$, the evaluation $C^\infty(U,\mathbb{K}) \times U \to \mathbb{K}$ is not continuous for the compact-open C^∞-topology.*

Proof. Suppose the evaluation is continuous at $(0,0)$. Then there are a compact $K \subseteq E$, $\delta > 0$ and an absolutely convex 0-neighborhood $W \subseteq E$ with $|f(x)| \le 1$ whenever $\sup_K |f| \le \delta$ and $x \in W$. By homogeneity, $K^\circ \subseteq \delta^{-1} W^\circ$, and by the bipolar theorem $W \subseteq \overline{W} = W^{\circ\circ} \subseteq \delta^{-1} K^{\circ\circ}$, where $K^{\circ\circ}$ is the closed absolutely convex hull of K. The absolutely convex hull of a precompact set is precompact, and so is its closure; hence E has a precompact 0-neighborhood and is finite-dimensional.

For the second statement, the compact-open C^∞-topology induces on $E' \subseteq C^\infty(U,\mathbb{K})$ the topology of E'_c (for a linear map all derivatives of order ≥ 1 are f itself or zero), and continuity of the evaluation at $(0,x_0)$ would give continuity of $(f,h) \mapsto f(x_0+h) - f(x_0) = f(h)$ at $(0,0)$. □

The proposition covers Banach and Hilbert spaces: there the evaluation $\mathcal{L}(E,F)_b \times E \to F$ *is* continuous [GN26, Rem. 2.3.9], but b is not the topology which the compact-open C^∞-topology induces on operators. For non-normable E there is no way out at all: the evaluation $\mathcal{L}(E) \times E \to E$ is discontinuous for *every* vector topology on $\mathcal{L}(E)$ (Maissen [Mai63, Satz 1], see also [KM97, p. 2] and the notes to Ch. 3 of [GN26]).

Remark 17.7 (The convenient calculus)*.* The convenient calculus of Frölicher, Kriegl and Michor [KM97] is cartesian closed. It works on Mackey complete (c^∞-complete) spaces, and this is where a completeness hypothesis enters. The price: the exponential law holds only as an isomorphism of bornological spaces (already for $E_1 = E_2 = \mathbb{R}$, the map Φ is discontinuous for the topologies of the convenient setting, see the notes to Ch. 1 of [GN26]), and convenient smooth maps need not be continuous. On Fréchet spaces both calculi have the same smooth maps.

17.4 The derivative as an operator-valued map

Let $f\colon U \to F$ be C^k with $k \ge 1$, $U \subseteq E$ open.

- *Into $\mathcal{L}(E,F)_c$, no hypotheses.* The map $x \mapsto \mathrm{d}f(x) \in \mathcal{L}(E,F)_c$ is C^{k-1}, for arbitrary locally convex E, F: apply the first part of Theorem 17.4 to $\mathrm{d}f$ and corestrict to the closed subspace $\mathcal{L}(E,F) \subseteq C(E,F)$ (as in the proof of [GN26, Prop. 2.3.17(b)]). Dually, for fixed x the map $C^\infty(U,F) \to \mathcal{L}(E,F)_c$, $f \mapsto \mathrm{d}f(x)$, is continuous linear, directly from the definition of the compact-open C^∞-topology.
- *Into $\mathcal{L}(E,F)_b$, normed spaces.* If E, F are normed, a Michal–Bastiani C^{k+1} map is Fréchet C^k (Theorem 8.1): continuity of $x \mapsto \mathrm{d}f(x)$ in operator norm costs one derivative, and nothing for smooth maps (Corollary 8.3). On the other hand, $f \mapsto \mathrm{d}f(x)$ is *not* continuous from $C^\infty(U,F)$ to $\mathcal{L}(E,F)_b$ when E is infinite-dimensional, since on E' the compact-open C^∞-topology is $c \ne b$.
- *Non-normable spaces.* By Maissen's theorem, there is no vector topology on $\mathcal{L}(E)$ making evaluation continuous, so no choice of topology turns the derivative into a well-behaved operator-valued map while keeping the chain rule continuous.

17.5 The convention adopted

To stay clear of all of the above, operator-valued data are always required to be smooth in uncurried form. A smooth family of projectors on an embedded submanifold $X \subseteq F$ is a

map $E\colon X \times F \to F$, linear in the second variable, such that $(p, w) \mapsto E_p w$ is smooth in charts (`LocallyConvex.Manifolds.ProjectorField`); the Christoffel symbols of a connection are smooth as functions $(x, a, b) \mapsto \Gamma_x(a, b)$ of three variables (`LocallyConvex.Manifolds.Connection`). No topology on $\mathcal{L}(F)$ is ever needed. On Banach spaces this loses nothing: by Corollary 8.3, $(x, w) \mapsto E_x w$ is then Fréchet smooth, hence so is $x \mapsto E_x$ into $\mathcal{L}(F)$ with the operator norm; the uncurried and the operator-norm notions of smoothness agree for C^∞ data.

The convention does have visible consequences on non-normable models: the dual bundle $T^*M = \bigcup_x (T_x M)'$ carries in general no canonical vector bundle structure [GN26, notes to Ch. 3], so 1-forms are treated as smooth maps on TM which are linear on the fibres; and the adjoint action of a Lie group is handled as the smooth map $G \times \mathfrak{g} \to \mathfrak{g}$, not as a map into $\mathcal{L}(\mathfrak{g})$.

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